

CHIRAL DYNAMICS

H. LEUTWYLER

*Institute for Theoretical Physics, University of Bern, Sidlerstr. 5,
CH-3012 Bern, Switzerland*

The effective field theory relevant for the analysis of QCD at low energies is reviewed. The foundations of the method are discussed in some detail and a few illustrative examples are described.

1 Introduction

The low energy properties of the strong interactions are governed by a chiral symmetry. For this reason, the physics of the degrees of freedom that are relevant in this domain is referred to as *chiral dynamics* and the corresponding effective field theory is called *chiral perturbation theory*. Effective field theories play an increasingly important role in physics, not only in the context of the strong interactions, but also in other areas: heavy quarks, electroweak symmetry breaking, spin models, magnetism, etc. In fact, one of the remarkable features of effective Lagrangians is their *universality*. A compound like La_2CuO_4 which develops two-dimensional antiferromagnetic layers and exhibits superconductivity up to rather high temperatures can be described by an effective field theory that closely resembles the one relevant for the strong interactions! The reason is that the effective theory only exploits the symmetry properties of the underlying theory and does not invoke the dynamics of the fields occurring therein. In fact, the development of chiral perturbation theory started in the 1960's, at a time when it was rather unclear whether the strong interactions could at all be described in terms of local fields. Let me briefly review the history of the subject.

In 1957, Goldberger and Treiman investigated the nucleon matrix element of the axial current.¹ The exchange of a pion between the nucleon and the current generates a pole at $t = M_\pi^2$, that is at a very small value of the momentum transfer. Assuming that the contribution from this nearby pole dominates the matrix element of the divergence at $t = 0$, they obtained the prediction

$$g_{\pi N\bar{N}} = \frac{g_A M_N}{F_\pi} \ ,$$

which determines the strength of the pion-nucleon interaction in terms of the axial current matrix element g_A , the nucleon mass M_N and the pion decay constant F_π . Inserting the experimental values available at that time, they

found that the relation is indeed obeyed at the 10 % level (in the meantime, the discrepancy has become significantly smaller: The current experimental information² indicates that the relation holds to within about 2 or 3%, but the value of $g_{\pi N \bar{N}}$ is still subject to sizeable uncertainties).

In 1960, Nambu then showed that the observed smallness of the pion mass can be explained on the basis of symmetry considerations.³ The argument relies on the fact that continuous symmetries may undergo spontaneous breakdown. If this happens, the spectrum of the theory necessarily contains massless particles, called Goldstone bosons, after Goldstone who established the implications of spontaneous symmetry breakdown in mathematically precise form.⁴ In the case of approximate symmetries, spontaneous breakdown gives rise to particles that are only approximately massless. According to Nambu, the pions are light because they are the Goldstone bosons of an approximate symmetry.

The significance of symmetries in particle physics was known since a long time. Twenty years earlier, Heisenberg had pointed out that the strong interactions are invariant under the group $SU(2)$ generated by isospin — if not exactly then to a high degree of accuracy.⁵ The relevance of *approximate symmetries*, however, only emerged in the beginning of the 1960's, mainly through the work of Gell-Mann.⁶ In particular, Gell-Mann and Ne'eman showed that the observed pattern of mesonic and baryonic states can be understood if one assumes the strong interactions to be approximately invariant under a larger group, $SU(3)$. The extended symmetry, termed the *eightfold way*, contains the isospin rotations as a subgroup. As the extra generators do not commute with the Hamiltonian, the corresponding currents are not conserved. For the Hamiltonian to be approximately symmetric, their divergence must, however, be small; accordingly, such currents were referred to as “partially conserved”.

The symmetry responsible for the smallness of the pion mass is of a different type. Since the pions carry negative parity, the relevant generators must change sign under space reflections: The corresponding partially conserved currents must be axial vectors. The assumption that the strong interactions admit Partially Conserved Axial vector Currents was termed the *PCAC hypothesis*.

Although the generators of an approximate symmetry group do not commute with the Hamiltonian, the commutators of the generators among themselves are fixed by the structure of the group, irrespective of symmetry breaking. The corresponding currents therefore obey a set of exact commutation relations — *current algebra*. In 1965/66 Adler and Weisberger⁷ established the first quantitative consequences of current algebra and PCAC, and Weinberg showed that the low energy properties of the amplitude describing the emission of any number of soft pions as well as the $\pi\pi$ scattering amplitude can unambiguously be predicted in this framework.^{8,9}

The method used to establish these results was based on an analysis of the Ward identities obeyed by the Green functions of the axial vector currents and was rather cumbersome. Weinberg, Wess, Zumino, Schwinger, Chang, Gursky, Lee and others, however, soon realized that the same results could be derived in a much simpler way, using *effective Lagrangians*. The general framework underlying this technique was analyzed by Callan, Coleman, Wess and Zumino¹⁰ in 1969. At about the same time, Dashen, Weinstein, Li and Pagels¹¹ started exploring the consequences of the fact that chiral symmetry is only an approximate symmetry, investigating the departures from the low energy theorems of current algebra due to symmetry breaking. A concise formulation in terms of the effective Lagrangian method was given by Weinberg¹² in 1979. In the meantime, QCD had been discovered, providing a coherent conceptual framework for an understanding of the strong interactions. In particular, this theory at once offered a natural explanation for the empirical fact that the Hamiltonian of the strong interactions exhibits an approximate symmetry: The Hamiltonian of QCD possesses an exact chiral symmetry if the quark masses are set equal to zero – hence an approximate one if these masses happen to be small.

The pioneering work on chiral dynamics concerned the properties of pion amplitudes on the mass shell. The key observation which gave birth to this development is that a suitable effective field theory involving Goldstone fields automatically generates on-shell amplitudes that obey the low energy theorems of current algebra and PCAC. The interaction among the Goldstone bosons is described by an effective Lagrangian that is invariant under global chiral transformations. The insight gained thereby not only led to a considerable simplification of current algebra calculations, but also paved the way to a systematic investigation of the low energy structure.

The shortcoming of the on-shell analysis is that it does not allow one to evaluate current matrix elements such as F_π or g_A , which – as illustrated by the Goldberger-Treiman relation – play an important role for the quantitative consequences of the symmetry properties. The problem originates in the fact that the on-shell analysis is based on *global* symmetry considerations. Global symmetry provides important constraints, but does not suffice to determine the low energy structure beyond leading order. A conclusive framework only results if the properties of the theory are analyzed off the mass shell: One needs to consider Green functions and study the Ward identities which express the symmetries of the underlying theory at the *local* level. The occurrence of anomalies illustrates the problem: Massless QCD is invariant under global $SU(N_f)_R \times SU(N_f)_L$, but – for more than two flavours – the corresponding effective Lagrangian is not.

In 1983, we proposed a method that incorporates the Ward identities *ab initio* and allows one to analyze the low energy structure of the Green functions in a controlled manner.^{13,14} We worked out a number of applications and estimated the effective couplings occurring at first nonleading order. Most of the more recent work on chiral dynamics is based on this framework.

The following presentation of chiral dynamics only covers a small fraction of the field. Throughout, I will restrict myself to the mesonic sector of Hilbert space. The extension of chiral perturbation theory to the sectors with nonzero baryon number has recently attracted considerable attention, for several reasons – the beautiful experimental results concerning pion photo- and electroproduction, the fact that the matrix element $\langle N | \bar{s}s | N \rangle$ can be determined by means of πN scattering, the Lorentz invariant formulation of the effective theory, baryons at large N_c , nuclear forces, to name a few. For a review of that development, I refer to the contributions by A. Manohar and U. Meissner in this volume. Even in the mesonic sector, the presentation is far from complete: The extension of the effective Lagrangian required to incorporate the degrees of freedom of the photons and leptons or to analyze nonleptonic weak transitions, will not be discussed. The significant progress in our understanding of the isospin breaking effects due to the mass difference between the u - and d -quarks – of relevance, for instance, for the analysis of the ratio ϵ'/ϵ – will not be covered, either. For a more complete picture of the current state of chiral dynamics, I refer to the review articles listed in the bibliography.¹⁵

2 Massless QCD – a theoretical paradise

For reasons that yet remain to be understood, it so happens that the Yukawa interaction of the u , d and s quarks with the Higgs field is weak, while the one of the remaining three quark flavours is strong. As a first approximation, we may consider the theoretical limit where m_u , m_d and m_s are set equal to zero, while m_c , m_b and m_t are sent to infinity. In this limit, QCD is a paradise of a theory: It does not contain a single dimensionless parameter. If the momenta are measured in units of the intrinsic scale of the theory, Λ_{QCD} , all of the transition probabilities of physical interest are unambiguously determined by the Lagrangian.

The Hamiltonian of QCD with three massless quark flavours is characterized by a high degree of symmetry, which originates in the fact that the interaction between the quarks and gluons is flavour independent and preserves helicity: It is invariant under independent flavour rotations of the right- and left-handed quark fields. The eight vector currents as well as the eight

axial currents

$$V_a^\mu = \bar{q}\gamma_\mu \frac{1}{2}\lambda_a q \ , \quad A_a^\mu = \bar{q}\gamma_\mu \gamma_5 \frac{1}{2}\lambda_a q \quad (1)$$

are conserved. The same holds for the singlet vector current V_0^μ , while the divergence of A_0^μ contains an anomaly,

$$\partial_\mu A_0^\mu = \sqrt{6}\omega \ , \quad \omega = \frac{1}{16\pi^2} \text{tr}_c G_{\mu\nu} \tilde{G}^{\mu\nu} \ , \quad \lambda_0 = \sqrt{\frac{2}{3}} \ . \quad (2)$$

The 9 conserved vector charges Q_0^V, \dots, Q_8^V and the 8 conserved axial charges Q_1^A, \dots, Q_8^A generate the group $G = \text{SU}(3)_R \times \text{SU}(3)_L \times \text{U}(1)_V$.

As shown by Vafa and Witten,¹⁶ the ground state of the theory is necessarily invariant under the subgroup generated by the vector charges: $Q_a^V |0\rangle = 0$. For the axial charges, however, there are two possibilities:

- a. $Q_a^A |0\rangle = 0$. The ground state is invariant under chiral rotations, G is realized as an ordinary Wigner-Weyl symmetry. The spectrum consists of degenerate multiplets that transform irreducibly under G and thus contain degenerate states of opposite parity.
- b. $Q_a^A |0\rangle \neq 0$. The ground state is not symmetric with respect to chiral rotations, G is realized as a spontaneously broken or Nambu-Goldstone symmetry. The spectrum consists of multiplets of the subgroup that leaves the vacuum invariant, $H = \text{SU}(3)_V \times \text{U}(1)_V$. One of these multiplets consists of the Goldstone bosons: Since the eight states $Q_a^A |0\rangle$ carry the same energy and momentum as the ground state, the spectrum must contain eight massless particles.

It still remains to be understood why the minimum of the energy occurs for an asymmetric rather than a symmetric state, so that alternative b. is realized in nature, but there is very strong experimental evidence for this to be the case. Indeed, the main features of the observed mass pattern are readily understood on this basis: The pion mass is small compared to the masses of all other hadrons. This is to be expected if the strong interactions possess an approximate, spontaneously broken symmetry with the pions as the corresponding Goldstone bosons. The Lagrangian of QCD does have the relevant approximate chiral symmetry, provided the quark masses m_u and m_d are small. As an immediate consequence, the strong interactions must approximately conserve isospin, because the corresponding symmetry breaking parameter, $m_u - m_d$, is then also small. The particle data tables show that the levels are grouped in multiplets of $\text{SU}(3)$. Since the splitting is much larger than the one within

the isospin multiplets, the symmetry breaking parameter $m_s - \frac{1}{2}(m_u + m_d)$ must be large compared to $m_u - m_d$. The observed pattern thus requires $m_s \gg m_d > m_u$. For the pions to be light and the eightfold way to be an approximate symmetry, m_u, m_d as well as $m_s - \frac{1}{2}(m_u + m_d)$ must be small. Hence the mass of the strange quark must be small, too – we must be living in a world that is close to the paradise described above.

In the real world, chiral symmetry is broken not only spontaneously, but also explicitly, through the quark mass term in the Hamiltonian,

$$H_{\text{QCD}} = H_0 + H_1 \quad , \quad H_1 = \int d^3x \, m_u \bar{u}u + m_d \bar{d}d + m_s \bar{s}s \quad . \quad (3)$$

Also, since the heavy quarks are not infinitely heavy, their degrees of freedom must be included in the Hamiltonian. As these are singlets under the group $\text{SU}(3)_R \times \text{SU}(3)_L \times \text{U}(1)_V$, they may be booked in H_0 – what counts for the low energy analysis of the theory is only that H_0 is invariant under this group.

QCD neatly explains why the pseudoscalar octet contains the eight lightest hadrons and why the mass pattern of this multiplet very strongly breaks eightfold way symmetry. These particles carry the quantum numbers of the Goldstone bosons required by the spontaneous breakdown of the approximate symmetry $\text{SU}(3)_R \times \text{SU}(3)_L \times \text{U}(1)_V \rightarrow \text{SU}(3)_V \times \text{U}(1)_V$. If the quarks were massless, M_π, M_K and M_η would vanish. The masses of the Goldstone bosons^a are due to the quark mass term in the Hamiltonian of QCD. For all other multiplets, the main contribution to the mass is given by the eigenvalue of H_0 – the quark mass term H_1 only generates a small perturbation that is responsible for the splitting of the levels, the state with the largest strange quark component winding up at the top. For the pseudoscalars, however, the main term is absent. First order perturbation theory shows that the square of the pion mass is given by $M_{\pi^+}^2 = (m_u + m_d)B + \dots$, where B is the matrix element of $\bar{u}u$ in the unperturbed state. For the kaon, the leading term involves the mass of the strange quark, $M_{K^+}^2 = (m_u + m_s)B + \dots$, $M_{K^0}^2 = (m_d + m_s)B + \dots$. The square of the kaon mass is about 13 times larger than the square of the pion mass because m_s is about 13 times larger than $m_u + m_d$. Eightfold way symmetry is perfectly consistent with the fact that m_u, m_d, m_s and hence M_π, M_K, M_η are very different from one another. In this context, the symmetry only implies that the matrix elements of the operators $\bar{u}u, \bar{d}d, \bar{s}s$ in the various states of the pseudoscalar octet are approximately determined by one and the same constant B .

^aSometimes, the name “Goldstone boson” is reserved for the case of an exact symmetry, replacing it by the term “Pseudo-Goldstone-boson” if the symmetry is an approximate one.

3 Effective action

I now turn to the Green functions of the various operators built with the quark fields: vector or axial currents, as well as scalar or pseudoscalar densities. These are conveniently collected in the effective action of the theory, which represents the response of the system to the perturbation generated by corresponding external fields. I denote the external field associated with the vector current by $v_\mu(x)$ and extend the Lagrangian by the term $\bar{q}\gamma^\mu v_\mu(x)q$. The field $v_\mu(x)$ is a matrix in flavour space, but is colour neutral and also commutes with the Dirac matrices. Similarly, the axial currents are generated by a term of the form $\bar{q}\gamma^\mu\gamma_5 a_\mu(x)q$. To include the scalar and pseudoscalar densities, it suffices to consider a space-dependent, complex quark mass matrix $m = m(x)$. For reasons which will become clear shortly, it is convenient to introduce a further external field $\theta(x)$, coupled to the winding number density ω defined in eq. (2). In the presence of these external fields, the QCD Lagrangian takes the form

$$\begin{aligned}\mathcal{L}_{\text{QCD}} &= \mathcal{L}_G + \bar{q}i\gamma^\mu(\partial_\mu - iG_\mu)q + \bar{q}\gamma^\mu(v_\mu + a_\mu\gamma_5)q - \bar{q}_R m q_L - \bar{q}_L m^\dagger q_R \\ \mathcal{L}_G &= -\frac{1}{2g^2}\text{tr}_c G_{\mu\nu}G^{\mu\nu} - \theta\omega \quad .\end{aligned}\tag{4}$$

The effective action of QCD is the logarithm of the corresponding vacuum-to-vacuum transition amplitude,

$$\exp i S_{\text{eff}}\{v, a, m, \theta\} = \langle 0 \text{ out} | 0 \text{ in} \rangle_{v, a, m, \theta}\tag{5}$$

and contains the various external fields as arguments. By construction, the quantities $v_\mu(x)$, $a_\mu(x)$ and $m(x)$ are $N_f \times N_f$ matrices acting in flavour space. While $v_\mu(x)$ and $a_\mu(x)$ are hermitean, the field $m(x)$ contains both a hermitean part, generating the scalar quark densities, and an antihermitean part, giving rise to the pseudoscalar operators.

The expansion of the effective action in powers of the external fields v_μ , a_μ , m and θ generates the Green functions of the massless theory. The quark condensate, for instance, is given by the term linear in $m(x)$, all other sources being switched off,

$$S_{\text{eff}} = - \int dx \langle 0 | \bar{q}_R m q_L + \bar{q}_L m^\dagger q_R | 0 \rangle + \dots\tag{6}$$

The various two-point functions are contained in the terms involving two external fields. In particular, the correlation function of the axial current is given by the term

$$S_{\text{eff}} = \frac{1}{2} i \int dx dy a_\mu^a(x) a_\nu^b(y) \langle 0 | T A_a^\mu(x) A_b^\nu(y) | 0 \rangle + \dots\tag{7}$$

where the field $a_\mu^b(x)$ represents the matrix $a_\mu(x)$ in the Gell-Mann basis, $a_\mu(x) = \frac{1}{2}\lambda_b a_\mu^b(x)$.

The same effective action also contains the Green functions of real QCD. To extract these, one considers the infinitesimal neighbourhood of the physical quark mass matrix m_0 rather than the vicinity of the point $m = 0$: Set $m(x) = m_0 + \tilde{m}(x)$ and treat $\tilde{m}(x)$ as an external field. The expansion of the effective action in powers of v_μ , a_μ , \tilde{m} and θ yields the Green functions of the vector, axial, scalar and pseudoscalar currents and of the operator $G_{\mu\nu}\tilde{G}^{\mu\nu}$ for the case of physical interest, where the quark masses are different from zero.

In path integral representation, the effective action of QCD is given by

$$\exp i S_{eff}\{v, a, m, \theta\} = \mathcal{N} \int [dG] \exp \left(i \int dx \mathcal{L}_G \right) \det D , \quad (8)$$

where D is the Dirac operator

$$D = i\gamma^\mu \{ \partial_\mu - i(G_\mu + v_\mu + a_\mu \gamma_5) \} - m \frac{1}{2}(1 - \gamma_5) - m^\dagger \frac{1}{2}(1 + \gamma_5) , \quad (9)$$

and \mathcal{N}^{-1} is the path integral for $v_\mu = a_\mu = m = \theta = 0$. In the present context, where the electroweak interactions are switched off, there is a sharp distinction between the colour field G_μ and the flavour fields v_μ, a_μ : While the former is a dynamical variable, which mediates the strong interactions and is to be integrated over in the path integral, the latter are classical auxiliary fields.

4 Chiral symmetry in terms of Green functions: Ward identities

The construction of the effective theory relies on the symmetry properties of the Green functions, more specifically, on the Ward identities obeyed by these. The Ward identities can be expressed as a remarkably simple property of the effective action: Disregarding the anomalies, S_{eff} is invariant under local chiral rotations. Since this property plays a central role in the following, I briefly show how it can be established.

Formally, the QCD Lagrangian is invariant under local $U(3)_R \times U(3)_L$ rotations of the quark fields,

$$q_R(x)' = V_R(x)q_R(x) , \quad q_L(x)' = V_L(x)q_L(x) , \quad V_R, V_L \in U(3) , \quad (10)$$

provided the external fields are transformed accordingly:

$$\begin{aligned} r_\mu(x)' &= V_R(x)r_\mu(x)V_R^\dagger - i\partial_\mu V_R V_R^\dagger , \\ l_\mu(x)' &= V_L(x)l_\mu(x)V_L^\dagger - i\partial_\mu V_L V_L^\dagger , \\ m(x)' &= V_R(x)m(x)V_L^\dagger , \end{aligned} \quad (11)$$

with $r_\mu = v_\mu + a_\mu$, $l_\mu = v_\mu - a_\mu$. As is well-known, however, only the modulus of the determinant of the Dirac operator is invariant under this operation – the phase picks up a change. The determinant is unique up to a local polynomial formed with the gluon and external fields. The polynomial may be chosen such that the determinant is invariant under the subgroup generated by the vector charges, as well as under gauge transformations of the gluon field. Under the infinitesimal transformation

$$V_R = \mathbf{1} + i\alpha + i\beta + \dots \quad , \quad V_L = \mathbf{1} + i\alpha - i\beta + \dots \quad ,$$

the change in the phase of the determinant then takes the form

$$\delta \ln \det D = -2i \int dx \langle \beta(x) \rangle \omega(x) - i \int dx \langle \beta(x) \rangle \Omega(x) \quad , \quad (12)$$

where $\langle X \rangle$ denotes the trace of the 3×3 matrix X . The gluon field only enters the first term, through the winding number density ω – this term gives rise to the U(1)-anomaly in the conservation law (2) for the singlet axial current. The second term only contains the external vector and axial fields,

$$\Omega = \frac{N_c}{4\pi^2} \epsilon^{\mu\nu\rho\sigma} \partial_\mu v_\nu \partial_\rho v_\sigma + \dots \quad (13)$$

The particular contribution indicated is the one that describes the anomalies in the Ward identities for the correlation function $\langle 0 | T A^\lambda V^\mu V^\nu | 0 \rangle$, which play a central role in the decay $\pi^0 \rightarrow \gamma\gamma$. The full expression for Ω also contains terms that are quadratic in a_μ , as well as contributions involving three or four vector or axial fields, which account for the anomalies in the Ward identities obeyed by the 4- and 5-point functions. The explicit expression for these terms is not relevant in our context, but it is essential that they are independent of the gluon field: This property implies that the second term in eq. (12) can be pulled out of the path integral. The first one can be absorbed with a change in the vacuum angle – this is the reason for introducing a term proportional to ω in the definition of the effective action. The net result is that the change generated by an infinitesimal chiral rotation of the external fields,

$$\begin{aligned} \delta v_\mu &= \partial_\mu \alpha + i[\alpha, v_\mu] + i[\beta, a_\mu] \quad , \quad \delta a_\mu = \partial_\mu \beta + i[\alpha, a_\mu] + i[\beta, v_\mu] \\ \delta m &= i(\alpha + \beta)m - im(\alpha - \beta) \quad , \quad \delta \theta = -2\langle \beta \rangle \quad , \end{aligned} \quad (14)$$

can be given explicitly: The effective action picks up the change

$$\delta S_{eff}\{v, a, m, \theta\} = - \int dx \langle \beta(x) \rangle \Omega(x) \quad . \quad (15)$$

The relation collects all of the Ward identities obeyed by the Green functions formed with the operators $\bar{q}\gamma_\mu\lambda q$, $\bar{q}\gamma_\mu\gamma_5\lambda q$, $\bar{q}\lambda q$, $\bar{q}i\gamma_5\lambda q$ and ω . It states that the effective action is gauge invariant under local chiral rotations, except for the anomalies, which are of purely geometric nature: The right-hand side of eq. (15) is independent of the coupling constant and of the quark masses – it only involves the number N_c of colours.

The relation $\delta\theta = -2\langle\beta\rangle$ specifies the transformation law of the vacuum angle only for infinitesimal chiral rotations. The one relevant for finite transformations is obtained by integrating this relation. As the group $U(1)$ is not simply connected, the result, however, is unique only up to multiples of 2π , so that only the transformation law for $e^{i\theta}$ is free of ambiguities:

$$e^{i\theta'} = \det(V_R^\dagger V_L) e^{i\theta} . \quad (16)$$

5 Basic low energy constants

Let us return to paradise and consider the quark condensate $\langle 0 | \bar{q}_R^\alpha q_L^\beta | 0 \rangle$ where $\alpha, \beta = 1, 2, 3$ indicates the quark flavour. Under left-handed chiral rotations, the operator $\bar{q}_R^\alpha q_L^\beta$ transforms according to the representation 3. If the ground state were invariant under these, the condensate would therefore vanish. In this sense, the matrix element $\langle 0 | \bar{q}_R^\alpha q_L^\beta | 0 \rangle$ represents a quantitative measure for the strength of the spontaneous symmetry breakdown and is referred to as an order parameter. The vacuum expectation value of any scalar operator that is not invariant under chiral transformations may serve as an order parameter. The quark condensate is the most important one, because it represents the one of lowest dimension. As there is no reason for the condensate to vanish, one generally assumes that it is different from zero. Lattice calculations provide some evidence for this to be the case, but it is notoriously difficult to explore the properties of the theory for small quark masses, not to speak of those of the massless theory.

Since the ground state is invariant under $SU(3)_V$, the condensate involves a single constant,

$$\langle 0 | \bar{q}_R^\alpha q_L^\beta | 0 \rangle = -\frac{1}{2}\delta^{\alpha\beta} C . \quad (17)$$

Moreover, invariance of the ground state under space reflections implies that $\langle 0 | \bar{q} i\gamma_5 q | 0 \rangle$ vanishes, so that C is real.

A nonzero condensate immediately implies that the spectrum contains massless particles. To see this, consider the correlation function of the axial and pseudoscalar current octets (to distinguish the octet components from the

singlets, I label these with the indices $i, k = 1, \dots, 8$, while $a, b = 0, \dots, 8$)

$$A_i^\mu = \bar{q} \gamma_\mu \gamma_5 \frac{1}{2} \lambda_i q \quad , \quad P_i = \bar{q} i \gamma_5 \frac{1}{2} \lambda_i q \quad ,$$

which obeys the Ward identity

$$\partial_\mu \langle 0 | T A_i^\mu(x) P_k(0) | 0 \rangle = -\frac{1}{4} i \delta(x) \langle 0 | \bar{q} \{ \lambda_i, \lambda_k \} q | 0 \rangle \quad . \quad (18)$$

The relation may be derived in a formal manner, by evaluating the derivative of the time-ordered product and using the equation of motion for the quark fields, but this is not without caveats – in general, operations of this sort are afflicted by ambiguities. The result may however be established in full rigour from the effective action. It suffices to consider the terms

$$\begin{aligned} S_{eff} = & - \int dx \langle 0 | \bar{q}_R m q_L + \bar{q}_L m^\dagger q_R | 0 \rangle \\ & - i \int dx dy \langle 0 | T \{ \bar{q} \gamma_\mu \gamma_5 a^\mu q \}_x \{ \bar{q}_R m q_L + \bar{q}_L m^\dagger q_R \}_y | 0 \rangle + \dots \end{aligned}$$

and to apply the infinitesimal chiral rotation (11) to the external fields $a_\mu(x)$, $m(x)$. The relation (15) requires the changes proportional to $\{\beta, m\}$, $\{\beta, m^\dagger\}$ in the first line to cancel those proportional to $\partial_\mu \beta(x) m(y)$, $\partial_\mu \beta(x) m^\dagger(y)$ in the second line. This condition indeed leads to eq. (18).

The above Ward identity can be solved explicitly. Lorentz invariance implies that the Fourier transform is of the form

$$\int dx e^{ip \cdot x} \langle 0 | T A_i^\mu(x) P_k(0) | 0 \rangle = p^\mu \Pi_{ik}^{AP}(p^2) \quad .$$

With the explicit representation (17) for the condensate, the identity (18) thus becomes

$$-p^2 \Pi_{ik}^{AP}(p^2) = \delta_{ik} C \quad . \quad (19)$$

Hence the function $\Pi_{ik}^{AP}(p^2)$ contains a pole at $p^2 = 0$ – the spectrum must contain massless particles. The result also shows that the correlation function under consideration is fully determined by the condensate:

$$\int dx e^{ip \cdot x} \langle 0 | T A_i^\mu(x) P_k(0) | 0 \rangle = \delta_{ik} \frac{p^\mu C}{-p^2 - i\epsilon} \quad . \quad (20)$$

The pole requires the existence of 8 massless one-particle states $|\pi^i\rangle$, with nonzero matrix elements

$$\langle 0 | A_i^\mu | \pi^k \rangle = i \delta_i^k p^\mu F \quad , \quad \langle 0 | P_i | \pi^k \rangle = \delta_k^i G \quad , \quad F G = C \quad . \quad (21)$$

In fact, only these intermediate states contribute and it is easy to see why that is so. The matrix element $\langle n|P_k|0\rangle$ vanishes unless the angular momentum of the state $|n\rangle$ vanishes. For such states, however, Lorentz invariance implies that the matrix element $\langle 0|A_i^\mu|n\rangle$ is proportional to the momentum p_n^μ of the state. At the same time, current conservation requires the matrix element to be transverse to p_n^μ . The two conditions can only be met if $p_n^2 = 0$, that is, if the vector $|n\rangle$ describes a massless particle of spin zero – a Goldstone boson.

The calculation demonstrates the validity of the Goldstone theorem in the context of QCD: The massless theory can have a nonzero quark condensate only if (a) the spectrum contains Goldstone bosons and (b) the corresponding one particle matrix elements of the axial and pseudoscalar currents are different from zero. The value of the pion matrix element of the axial current, the pion decay constant $F_\pi = 92.4 \text{ MeV}$, is known experimentally, from the decay $\pi \rightarrow \mu\nu$. Since the Goldstone bosons of QCD are the pseudoscalar mesons, this constant must approach a nonzero limit F when the quark masses are sent to zero.

The Gell-Mann-Oakes-Renner relation¹⁷ is an immediate consequence: For nonzero quark masses, the divergence of the axial current is related to the pseudoscalar density by $\partial_\mu(\bar{u}\gamma^\mu\gamma_5 d) = (m_u + m_d)\bar{u}i\gamma_5 d$. The vacuum-to-pion matrix element of this equality shows that the physical one particle matrix elements obey the exact relation $M_\pi^2 F_\pi = (m_u + m_d)G_\pi$. The coefficient of the leading term in the expansion of the pion mass in powers of the quark masses, $M_\pi^2 = (m_u + m_d)B + \dots$, is therefore determined by the condensate and by the pion decay constant: $B = G/F = C/F^2$.

The above relations involve two independent low energy constants, F and B . In particular, the value of the quark condensate in the massless theory may be expressed in terms of these: $\langle 0|\bar{u}u|0\rangle = -F^2 B$. In fact, the same two constants fully determine the leading low energy singularities in all of the Green functions. In the case of the correlation function $\langle 0|TA_i^\mu(x)A_k^\nu(0)|0\rangle$, for instance, this can be seen as follows. In the massless theory, this function obeys the Ward identity $\partial_\mu\langle 0|TA_i^\mu(x)A_k^\nu(0)|0\rangle = 0$, so that the Fourier transform is transverse to the momentum,

$$i\int dx e^{ip\cdot x} \langle 0|TA_i^\mu(x)A_k^\nu(0)|0\rangle = (p^\mu p^\nu - g^{\mu\nu}p^2)\delta_{ik}\Pi^{\text{AA}}(p^2) \quad . \quad (22)$$

The one-particle intermediate states generate a pole at $p^2 = 0$, whose residue is also determined by the constant F :

$$\Pi^{\text{AA}}(p^2) = \frac{F^2}{-p^2 - i\epsilon} + \dots \quad (23)$$

6 Low energy expansion of the effective action

The result obtained in the preceding section for the quark condensate and for the correlation functions of the operators A_i^μ , P_i amounts to the following explicit expression for the relevant terms in the effective action:

$$\begin{aligned} S_{eff} = & \int dx \frac{1}{2} F^2 B \langle m(x) + m^\dagger(x) \rangle \\ & - \int dxdy \frac{1}{2} i F^2 B \sum_{i=1}^8 \partial^\mu a_\mu^i(x) \Delta_0(x-y) \langle \lambda_i(m(y) - m^\dagger(y)) \rangle \\ & - \int dxdy \frac{1}{4} F^2 \sum_{i=1}^8 a_{\mu\nu}^i(x) \Delta_0(x-y) a^{\mu\nu i}(x) + \dots \end{aligned} \quad (24)$$

The function $\Delta_0(x)$ stands for the massless propagator,

$$\Delta_0(x) = \frac{1}{(2\pi)^4} \int dp \frac{e^{-ip \cdot x}}{-p^2 - i\epsilon} = \frac{i}{4\pi^2(-x^2 - i\epsilon)} ,$$

and $a_{\mu\nu}^i$ is the field strength, $a_{\mu\nu}^i = \partial_\mu a_\nu^i - \partial_\nu a_\mu^i$.

The expression shows that the effective action of massless QCD is an inherently nonlocal object – the exchange of Goldstone bosons implies that the correlation functions only drop off with a power of the distance. This is in marked contrast to the effective action of Heisenberg and Euler, who considered the correlation functions of the current for electrons exposed to an external electromagnetic field.¹⁸ The corresponding effective action is given by the sum of all one-loop graphs containing an arbitrary number of external field vertices,

$$\exp i S_{eff}^e\{A\} = \det \{i\gamma^\mu(\partial_\mu - ieA_\mu) - m_e\} . \quad (25)$$

In this case, the correlation functions drop off exponentially with the distance. If the external field is weak ($|F_{\mu\nu}| \ll m_e^2$, $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$) and varies only slowly, so that the typical wavelengths are small compared to the Compton wavelength of the electron ($|\partial_\lambda F_{\mu\nu}| \ll m_e |F_{\mu\nu}|$), the effective action may be expanded in powers of derivatives:

$$\begin{aligned} S_{eff}^e\{A\} = & \int dx \mathcal{L}_{eff} , \\ \mathcal{L}_{eff} = & -\frac{e^2 \Pi^{jj}(0)}{4} F^{\mu\nu} F_{\mu\nu} - \frac{e^2}{240\pi^2 m_e^2} \partial^\lambda F^{\mu\nu} \partial_\lambda F_{\mu\nu} \\ & - \frac{e^2}{2240\pi^2 m_e^4} \square F^{\mu\nu} \square F_{\mu\nu} + \frac{e^4}{1440\pi^2 m_e^4} \{(F^{\mu\nu} F_{\mu\nu})^2 + 7(F^{\mu\nu} \tilde{F}_{\mu\nu})^2\} + \dots \end{aligned} \quad (26)$$

The contributions quadratic in the field strength are described by the vacuum polarization $\Pi^{jj}(p^2)$ – the analogue of the quantity $\Pi^{AA}(p^2)$ introduced in eq. (22), the axial current being replaced by the electromagnetic one. The various terms arise from the Taylor series expansion of this function in powers of p^2/m_e^2 . The first term in the Heisenberg-Euler Lagrangian is proportional to the Lagrangian of the free electromagnetic field and merely renormalizes the photon wave function, $A_\mu^{ren} = \{1 + \frac{1}{2}e^2\Pi^{jj}(0)\}A_\mu$. The remainder amounts to a modification of the Maxwell Lagrangian and deforms the electromagnetic field generated by a given charge distribution. For slowly varying fields, the effect is dominated by the contribution of order $(\partial_\lambda F_{\mu\nu})^2$, referred to as the Uehling term. In particular, this term generates a small contribution to the Lamb shift (spacing between the S - and P -wave bound states of the hydrogen atom with principal quantum number $n = 2$).

The origin of the qualitative difference between the two effective actions is evident: The spectrum of the states that can be created by an external electromagnetic field has an energy gap, $\Delta E = 2m_e$, while the spectrum of massless QCD does not. External fields can generate Goldstone bosons, even if their wavelength is large. In reality, QCD also has an energy gap: $\Delta E = M_\pi$. Accordingly, for external fields that vary only slowly on the scale set by the Compton wavelength of the pion, the effective action also admits a derivative expansion that consists of a string of local terms. The range of validity of such a representation, however, is very limited, because m_u , m_d and hence the pion mass are small. If the quark masses are set equal to zero, the straightforward derivative expansion becomes entirely meaningless.

In contrast to the case of the correlation function $\langle 0|TA_i^\mu P_k|0\rangle$, which exclusively receives a contribution from the exchange of single Goldstone bosons, the quantity $\langle 0|TA_i^\mu A_k^\nu|0\rangle$ also picks up contributions from spin 1 intermediate states with $3, 5, \dots$ pseudoscalar mesons, which the representation for the effective action in eq. (24) does not account for. In the function $\Pi^{AA}(p^2)$, these generate a branch cut starting at $p^2 = 0$. At low energies, phase space strongly suppresses the discontinuity across the cut, but in the vicinity of the resonance $a_1(1260)$, there is a pronounced peak. The pole term dominates in the sense that the remainder approaches a finite limit H when $p \rightarrow 0$. We may view the pole as the leading term in the expansion of $\Pi^{AA}(p^2)$ in powers of the momentum:

$$\Pi^{AA}(p^2) = \frac{F^2}{-p^2 - i\epsilon} + H + O(p^2) \quad . \quad (27)$$

The constant H gives rise to an additional contribution in the expression for the effective action: a local term proportional to $\int dx H \sum_i (a_{\mu\nu}^i)^2$, which resembles

the leading term in the Heisenberg-Euler action. The systematic analysis of the effective theory, which will be sketched below, orders the contributions according to powers of the momentum and automatically accounts for the above term at first nonleading order of this “low energy expansion”. The imaginary part generated by intermediate states with 3 Goldstone bosons manifests itself at next-to-next-to leading order of that expansion. The resulting representation for the effective action in effect also amounts to a derivative expansion. It is of a nonlocal type, because the expansion contains inverse powers of the momentum.

The pole term provides an adequate approximation of the function $\Pi^{\text{AA}}(p^2)$ only for momenta that are small compared to the mass of the $a_1(1260)$. In the case of the correlation function of the vector current, where the $\rho(770)$ generates a peak in the imaginary part, the domain where the first one or two terms in the expansion in powers of the momentum provide a decent approximation is even smaller. Generally speaking, all of the momenta must be small compared to the intrinsic scale of QCD. The quantitative form of this condition depends on the channel under study. The internal consistency of the effective theory also leads to a constraint on the magnitude of the momenta: As will be discussed below, the loop graphs generate contributions at next-to-leading order. These must be small compared to the leading terms, a requirement that only holds if the momenta obey the condition $p \ll 4\pi F_\pi / \sqrt{N_f} \simeq 700 \text{ MeV}$.

7 Effective Lagrangian

The expression (24) is reminiscent of the effective action of a free field theory that involves massless scalar fields. Indeed, we can introduce eight pseudoscalar fields $\pi^1(x), \dots, \pi^8(x)$ and consider the Lagrangian

$$\begin{aligned} \mathcal{L}_{\text{eff}} &= \frac{1}{4} \langle d_\mu \pi d^\mu \pi \rangle + \frac{1}{2} F^2 B \langle m + m^\dagger \rangle - \frac{1}{2} i F B \langle \pi m - \pi m^\dagger \rangle , \\ \pi &= \sum_{i=1}^8 \pi^i \lambda_i , \quad d_\mu \pi = \sum_{i=1}^8 (\partial_\mu \pi^i - F a_\mu^i) \lambda_i , \end{aligned} \quad (28)$$

which is quadratic in these fields, so that the effective action coincides with the classical action. The corresponding equation of motion reads:

$$\square \pi^i(x) = F \partial^\mu a_\mu^i(x) - i F B \langle \lambda_i \{ m(x) - m^\dagger(x) \} \rangle . \quad (29)$$

It is straightforward to work out the classical action of this model and to check that the result of this calculation within classical field theory does reproduce the terms in eq. (24). The same calculation also yields the pole term in the correlation function $\langle 0 | T P_i(x) P_k(0) | 0 \rangle$, which is not included there.

The part of the effective action of massless QCD that we have picked out may thus equally well be described in terms of an entirely different field theory, which instead of quarks and gluons contains a set of pseudoscalar fields as dynamical variables. In contrast to the effective action, the relevant Lagrangian does represent a local expression. It is convenient to count the external field $a_\mu^i(x)$ as a quantity of the same order as the derivative, $O(a) = O(\partial) = O(p)$ and to book the field $m(x)$ as a term of $O(p^2)$. The above expression for the Lagrangian then represents a term of $O(p^2)$.

It is clear, however, that this framework is incomplete: (a) it does not cover all of the Green functions of QCD and (b) the corresponding representation for the correlation function $\langle 0 | T A_i^\mu(x) A_k^\nu(0) | 0 \rangle$ only accounts for the contribution that dominates at low energies. As discussed below, both of these limitations can be removed, at least in principle: If the effective Lagrangian is chosen properly, the resulting effective action coincides with the one of QCD, to any desired order in the low energy expansion. In this sense, there exists an alternative, exact representation of QCD: The path integral in eq. (8) can be replaced by a path integral over the effective fields $\pi^1(x), \dots, \pi^8(x)$,

$$\exp i S_{eff}\{v, a, m, \theta\} = \mathcal{N}_{eff} \int [d\pi] \exp(i \int dx \mathcal{L}_{eff}) \quad , \quad (30)$$

where \mathcal{N}_{eff}^{-1} is the same path integral evaluated at $v_\mu = a_\mu = m = \theta = 0$. The full effective Lagrangian is a local expression of the form

$$\mathcal{L}_{eff} = \mathcal{L}_{eff}(\pi, v, a, m, \theta; \partial\pi, \partial v, \partial a, \partial m, \partial\theta; \partial^2\pi, \dots) \quad .$$

It can be ordered by counting the number of fields and derivatives with

$$\{\pi, \theta\} = O(1) \quad , \quad \{\partial, v, a\} = O(p) \quad , \quad m = O(p^2) \quad . \quad (31)$$

Lorentz invariance permits only even orders. The expansion starts at $O(p^2)$:

$$\mathcal{L}_{eff} = \mathcal{L}^{(2)} + \mathcal{L}^{(4)} + \mathcal{L}^{(6)} + \dots \quad (32)$$

For a detailed proof of this claim, I refer to the literature.¹⁹ In the following, I first briefly describe the main properties of the effective Lagrangian and of the path integral (30) and then make a few comments about the proof of the statement that the effective theory reproduces the Green functions of QCD, order by order in the low energy expansion.

8 Effective field theory

The leading term in the derivative expansion of \mathcal{L}_{eff} is the Lagrangian of the nonlinear σ -model:

$$\mathcal{L}^{(2)} = \frac{1}{4} F^2 \langle \nabla_\mu U \nabla^\mu U^\dagger \rangle + \frac{1}{2} F^2 B \langle m U^\dagger + U m^\dagger \rangle + \frac{1}{12} H_0 D_\mu \theta D^\mu \theta \quad .$$

The effective field is described in terms of a unitary 3×3 matrix, $UU^\dagger = \mathbf{1}$. The covariant derivatives stand for

$$\nabla_\mu U = \partial_\mu U - i(v_\mu + a_\mu)U + iU(v_\mu - a_\mu), \quad D_\mu \theta = \partial_\mu \theta + 2\langle a_\mu \rangle.$$

With the parametrization $U(x) = \exp i\pi(x)/F$, one readily checks that $\mathcal{L}^{(2)}$ indeed contains the effective Lagrangian given in eq. (28): That part accounts for the first three terms in the expansion in powers of the field $\pi(x)$, for $v_\mu = \langle a_\mu \rangle = \theta = 0$. The expansion does not stop there, however. The nonlinear σ -model Lagrangian also contains vertices describing interactions among the Goldstone bosons, and contributions that involve the external fields $v_\mu, \langle a_\mu \rangle$ and θ : This Lagrangian also accounts for the leading terms in the low energy expansion of the scattering amplitudes and of the correlation functions of the operators V_a^μ, A_0^μ, ω . In fact, all of the predictions obtained in the sixties, on the basis of current algebra and PCAC, can be worked out in a comparatively very simple manner with this Lagrangian. Apart from the term $H_0 D_\mu \theta D^\mu \theta$, which exclusively generates a contact contribution in the two-point functions of the operators A_0^μ and ω , the Lagrangian $\mathcal{L}^{(2)}$ only involves the two basic low energy constants F and B introduced in section 5. In the large N_c limit, F is of order $\sqrt{N_c}$, while B and H_0 are of order 1.

The crucial property that distinguishes the nonlinear σ -model from all other field theory models with eight scalar or pseudoscalar fields as dynamical variables is that it is manifestly invariant under the group $U(3)_R \times U(3)_L$ of local chiral rotations: The transformation (11), (16) of the external fields leaves the Lagrangian $\mathcal{L}^{(2)}$ invariant, provided the meson field $U(x)$ is transformed with

$$U(x)' = V_R(x)U(x)V_L^\dagger(x) \quad . \quad (33)$$

Note that this transformation law does not leave the determinant of $U(x)$ invariant. Indeed, $U(x)$ is an element of $SU(3)$ only for $\theta = 0$. The θ -term in the Lagrangian of QCD – which is needed to analyze the consequences of chiral symmetry for the full group $U(3)_R \times U(3)_L$ of chiral rotations – modifies the condition $\det U = 1$ that pertains to the standard nonlinear σ -model: The condition is replaced by the constraint

$$\det U = e^{-i\theta} \quad , \quad (34)$$

which, in view of eq. (16), is consistent with the transformation law (33).

I add a remark of technical nature. In the presence of the singlet external fields θ and $\langle a_\mu \rangle$, the covariant derivative $\nabla_\mu U$ is not convenient to work with, because the trace $\langle U^\dagger \nabla_\mu U \rangle$ does not vanish. In the following, I instead use $D_\mu U = \nabla_\mu U + \frac{1}{3} i D_\mu \theta U$, which does obey $\langle U^\dagger D_\mu U \rangle = 0$. In this notation, the leading term of the effective Lagrangian takes the form

$$\begin{aligned} \mathcal{L}^{(2)} &= \frac{1}{4} F^2 \langle D_\mu U D^\mu U^\dagger \rangle + \frac{1}{2} F^2 B \langle m U^\dagger + U m^\dagger \rangle + \frac{1}{12} \tilde{H}_0 D_\mu \theta D^\mu \theta , \\ D_\mu U &= \partial_\mu U - i (v_\mu + a_\mu) U + i U (v_\mu - a_\mu) + \frac{1}{3} i D_\mu \theta U , \end{aligned} \quad (35)$$

with $\tilde{H}_0 = H_0 + F^2$. Note that, in the large N_c limit, \tilde{H}_0 is of $O(N_c)$ – the leading term in the $1/N_c$ expansion of this constant is given by F^2 .

The invariance of $\mathcal{L}^{(2)}$ immediately implies that the corresponding classical action is invariant under local chiral transformations of the external fields. The classical action collects the set of all tree graph contributions to the path integral, so that this part of the effective action is invariant. The loop graphs cannot simply be dropped – otherwise, unitarity is violated – but they represent contributions of nonleading order:¹² In dimensional regularization, those graphs of $\mathcal{L}^{(2)}$ that contain ℓ meson loops represent contributions of order $p^{2\ell+2}$. Accordingly, at leading order of the low energy expansion, only the tree graphs matter. The claim that the effective action of QCD can be represented as a path integral over meson fields thus implies that the leading contributions in the expansion of this effective action are given by the classical action of the nonlinear σ -model and are therefore invariant under chiral rotations. This conclusion is in agreement with the fact that the anomalous terms occurring in the Ward identities represent contributions of order p^4 : As indicated by the expression in eq. (13), the quantity Ω represents a term of that order.

9 Illustration: Topological susceptibility

At low energies, the leading contributions to the Green functions of QCD are given by the tree graphs of $\mathcal{L}^{(2)}$. I illustrate the content of this claim with the topological susceptibility,^b

$$\chi(q^2) = i \int dx e^{iq \cdot x} \langle 0 | T \omega(x) \omega(0) | 0 \rangle . \quad (36)$$

Because the dependence of the susceptibility on the quark masses is very interesting, I do not set m_u equal to m_d . To evaluate the corresponding low

^bThe sign convention adopted here is such that the imaginary part of $\chi(q^2)$ is positive. In this convention, $\chi(0)$ is negative: The mean square winding number per unit volume of euclidean space is given by $-\chi(0)$.

energy representation at leading order, it suffices to calculate the extremum of the classical action of $\mathcal{L}^{(2)}$ in the presence of the external field $\theta(x)$, while $m(x)$ is identified with the physical quark mass matrix and all other external fields are switched off. The correlation function of interest is the coefficient of the term quadratic in $\theta(x)$. The classical equation of motion implies that all components of $\pi(x)$ vanish at the extremum, except $\pi^3(x)$ and $\pi^8(x)$. The quadratic part of the Lagrangian yields the corresponding masses. The states π^0 and η mix and the levels repel. The eigenvalues are

$$\begin{aligned} M_{\pi^0}^2 &= (m_u + m_d)B - \Delta \quad , \quad M_\eta^2 = \frac{2}{3}(\hat{m} + 2m_s)B + \Delta \quad , \\ \Delta &= \frac{4 \sin^2 \epsilon}{3 \cos 2 \epsilon} (m_s - \hat{m})B \quad , \quad \text{tg } 2 \epsilon = \frac{\sqrt{3}}{2} \frac{m_d - m_u}{m_s - \hat{m}} \quad . \end{aligned}$$

The susceptibility is obtained by solving the classical equation of motion to first order in θ . The result reads:¹⁴

$$\begin{aligned} \chi(q^2) &= \sum_{P=\pi^0, \eta} \frac{|\langle 0 | \omega | P \rangle|^2}{M_P^2 - q^2} - \frac{1}{9} BF^2(m_u + m_d + m_s) + \frac{1}{6} \tilde{H}_0 q^2 + O(p^4) \quad , \\ \langle 0 | \omega | \pi^0 \rangle &= \frac{1 - \frac{4}{3} \sin^2 \epsilon}{2 \cos \epsilon} (m_d - m_u) BF \quad , \\ \langle 0 | \omega | \eta \rangle &= \frac{2(1 - 4 \sin^2 \epsilon) \cos \epsilon}{3\sqrt{3} \cos 2 \epsilon} (m_s - \hat{m}) BF \quad . \end{aligned}$$

The quantity $\chi(0)$ represents the second derivative of the vacuum energy with respect to θ and must vanish if either m_u , m_d or m_s are sent to zero, because the θ -dependence then disappears.²⁰ Indeed, the pole contribution in the above representation cancels the momentum independent term if one of the quark masses is turned off. The result takes the simple form:²¹

$$\chi(0) = -BF^2 m_{red} + O(m^2) \quad , \quad (37)$$

where m_{red} stands for the reduced mass,

$$\frac{1}{m_{red}} = \frac{1}{m_u} + \frac{1}{m_d} + \frac{1}{m_s} \quad . \quad (38)$$

The relation amounts to a low energy theorem for the mean square winding number per unit volume: The expansion of this quantity in powers of the quark masses starts with $\langle \nu^2 \rangle / V = BF^2 m_{red} + O(m^2)$.

The first derivative $\chi'(0)$ is of interest in connection with the spin content of the proton.^{22,23} The explicit expression reads:

$$\chi'(0) = \frac{1}{2} F^2 m_{red}^2 \left\{ \frac{1}{m_u^2} + \frac{1}{m_d^2} + \frac{1}{m_s^2} \right\} + \frac{1}{6} H_0 + O(m) \quad . \quad (39)$$

As mentioned earlier, the second term is suppressed as compared to the first: $F^2 = O(N_c)$, $H_0 = O(1)$. In the following, I only consider the leading contribution, which is fully determined by the quark mass ratios $m_u : m_d : m_s$.

Numerically, inserting the phenomenological values of the mass ratios,²⁴ and using $F \simeq F_\pi = 92.4 \text{ MeV}$, the formula predicts $\chi'(0) = 2.2 \cdot 10^{-3} \text{ GeV}^2$, in remarkable agreement with the sum rule determination of Ioffe and collaborators,²³ who find $\chi'(0) = (2.3 \pm 0.6) \cdot 10^{-3} \text{ GeV}^2$ and $(2.0 \pm 0.5) \cdot 10^{-3} \text{ GeV}^2$, depending on the method used. Stated differently, the sum rule results confirm the validity of the Okubo-Iizuka-Zweig rule in this case: The contribution from the term H_0 is numerically small.

Note that the above algebraic result is extremely sensitive to the pattern of chiral symmetry breaking: In view of $m_u, m_d \ll m_s$, the formula roughly yields $\chi'(0) \simeq \frac{1}{2} F^2 (m_u^2 + m_d^2) / (m_u + m_d)^2$, so that the result is nearly independent of m_s , but changes by a factor of 2 if the ratio m_u/m_d is varied between 0 and 1. The leading term in the quark mass expansion of $\chi'(0)$ is of a similar structure as the one in the mass ratio $(M_{K^0}^2 - M_{K^+}^2) / M_\pi^2 \simeq (m_d - m_u) / (m_u + m_d)$. The connection with the proton spin content indicates that a similar sensitivity to the ratio m_u/m_d also occurs in the relevant nucleon matrix elements.

In principle, one could also derive the above results by means of current algebra and PCAC, but the calculation would be considerably more tedious. Beyond leading order, it is practically impossible to study such quantities without making use of the effective Lagrangian method.

I add a comment²⁵ concerning the definition of $\chi(q^2)$. The correlation function $\langle 0 | T \omega(x) \omega(0) | 0 \rangle$ is too singular for the integral in eq. (36) to make sense as it stands. The corresponding dispersion relation contains two subtractions:²⁶

$$\chi(q^2) = \chi(0) + q^2 \chi'(0) + \frac{q^4}{\pi} \int \frac{ds}{s^2(s - q^2 - i\epsilon)} \text{Im } \chi(s) \ .$$

The first one of these is fixed by the invariance of the effective action under local chiral rotations, but the second is not: (i) For the path integral over the quarks and gluons to make sense in the presence of external fields, all terms of mass dimension ≤ 4 that are consistent with the symmetries of the theory must be included in the QCD Lagrangian. (ii) The invariance property (15) excludes a term proportional to θ^2 , but does allow one of the form $h_0 D_\mu \theta D^\mu \theta$. In fact, such a term is needed to absorb the quadratic divergences occurring in the perturbation theory graphs relevant for the topological susceptibility. Hence the value of $\chi'(0)$ depends on the method used when subtracting these infinities. How come that we can make a statement about its numerical magnitude ?

The reason is that all of the graphs contributing to the correlation function either remain finite or disappear when N_c is sent to infinity. The same is true of

the renormalization ambiguity contained therein: On the level of the effective theory, the constant $h_0 = O(1)$ exclusively contributes to $H_0 = O(1)$. Since $\chi'(0)$ is a quantity of $O(N_c)$, the problem only shows up at nonleading orders of the $1/N_c$ expansion. The formula (39) shows that the leading term in the simultaneous expansion of $\chi'(0)$ in powers of m_u, m_d, m_s and $1/N_c$ is fully determined by the quark mass ratios and by the pion decay constant. In the evaluation of the sum rules, the problem does not show up, because the perturbative contributions that would require renormalization are discarded – the results obtained concern the nonperturbative contributions to $\chi'(0)$.

10 Higher orders

At next-to-leading order, the Lagrangian involves further effective coupling constants. The effective theory contains infinitely many such constants, which chiral symmetry leaves undetermined – they represent the analogues of the Taylor coefficients occurring in the Heisenberg-Euler Lagrangian. There is a difference in that those coefficients can explicitly be calculated in terms of the electron mass, while an explicit expression for the effective coupling constants F, B, \dots in terms of the scale of QCD is not available. Quite a few of these have been determined on the basis of experimental information and for some, a numerical determination on the lattice has been performed.²⁷

As mentioned above, the one-loop graphs of $\mathcal{L}^{(2)}$ represent contributions of order p^4 . Dimensional regularization preserves the symmetries of the Lagrangian. The divergences arising from the one-loop graphs thus represent local terms that are invariant under $U(3)_R \times U(3)_L$. Since the Lagrangian $\mathcal{L}^{(4)}$ contains all terms permitted by the symmetry, the divergences contained in the one-loop graphs of $\mathcal{L}^{(2)}$ can be absorbed in a renormalization of the effective coupling constants in $\mathcal{L}^{(4)}$. The argument extends to graphs with an arbitrary number of loops, including those that involve vertices from the higher order terms in the derivative expansion of the effective Lagrangian.

Taken by itself, the nonlinear σ -model does not make sense, because the divergences generated by the quantum fluctuations cannot be absorbed by renormalizing the coupling constants F, B . Indeed, the model only represents the leading term in the derivative expansion of the full effective Lagrangian. The effective theory does provide the proper embedding for the model to be meaningful: The divergences can be absorbed in the couplings of higher order. The results obtained on the basis of the effective theory to any given order in the low energy expansion are unambiguous. In particular, they do not depend on the regularization used – in this sense, the effective theory represents a renormalizable framework that involves infinitely many coupling constants.

Finally, I recall that the full effective action of QCD is not invariant under chiral rotations, because of the anomalies. For the effective theory to account for the anomalous terms in the Ward identities, the effective Lagrangian must contain contributions that are not invariant. In fact, a closed expression for the relevant contributions is known since a long time: the Wess-Zumino-Witten Lagrangian.²⁸ The full effective Lagrangian is obtained by first writing down all possible vertices that are invariant under $U(3)_R \times U(3)_L$ and then adding this term, which represents a contribution of $O(p^4)$ and thus belongs to $\mathcal{L}^{(4)}$. The WZW-term does not involve any free parameters – like the anomalies themselves, it represents a purely geometric contribution that is fully determined by the number of colours.

11 Outline of the proof

I now briefly sketch the proof¹⁹ of the claim that (a) the low energy expansion of the effective action of QCD can be worked out by means of an effective field theory and (b) the relevant effective Lagrangian is invariant under local chiral rotations, except for the WZW-term. The basic hypothesis underlying this proof is that the Goldstone bosons required by spontaneous symmetry breakdown are the only massless particles contained in the spectrum of asymptotic states, so that only these generate singularities at low energies.

One first shows that, as a consequence of the Ward identities, Goldstone bosons of zero momentum cannot interact. This is crucial, because it implies that the interaction becomes weak at low energies – the reason why the low energy properties of QCD can be worked out explicitly, despite the fact that the interaction among the quarks and gluons is strong there.

Next, one considers the Goldstone boson scattering amplitude. The structure of the low energy singularities contained therein is determined by the cluster decomposition: One-particle exchange generates poles, while the exchange of two or more particles produces branch cuts. Since Goldstone bosons of zero momentum do not interact, the scattering amplitude disappears if all of the momenta are sent to zero. Unitarity implies that the imaginary part of the scattering amplitude is given by its square. In four dimensions, the relevant phase space factors yield nonnegative powers of the momentum, so that the imaginary part disappears more rapidly than the real part when the momenta tend to zero: The contributions generated by the exchange of two or more particles only show up at nonleading orders of the low energy expansion. The leading term exclusively contains the poles due to one-particle exchange. At leading order, their residues – the one-particle irreducible parts – are free of singularities and can thus be expanded in the momenta. This property under-

lies all of the early work on current algebra and PCAC and used to be referred to as the pion pole dominance hypothesis.

The one-particle irreducible parts vanish at zero momentum. Lorentz invariance and Bose statistics thus imply that the leading term in their low energy expansion is of the form

$$- \sum_{\text{perm}(1, \dots, n)} \left\{ g_{i_1 \dots i_n} p_1 \cdot p_2 + h_{i_1 \dots i_n} p_1^2 \right\} ,$$

where p_1, \dots, p_n are the momenta flowing into the irreducible part in question and i_1, \dots, i_n label the flavour quantum numbers of the corresponding Goldstone bosons. The generic form of the leading contributions in the low energy expansion of the scattering amplitude is given by a product of pole terms and factors of the above form. Since contributions proportional to the square of the momentum of one of the particles cancel against the corresponding pole term, we may without loss of generality set $h_{i_1 \dots i_n} = 0$.

At leading order of the low energy expansion, the scattering amplitude is of the same structure as the tree graphs of a field theory. In fact, the tree graphs of the Lagrangian

$$\mathcal{L}^{(2)} = \sum_{i, k} g_{ik}(\pi) \partial_\mu \pi^i \partial^\mu \pi^k, \quad g_{ik}(\pi) = \frac{1}{2} \delta_{ik} + \sum_{n=3}^{\infty} \sum_{i_3, \dots, i_n} g_{ik i_3 \dots i_n} \pi^{i_3} \dots \pi^{i_n}$$

generate precisely the same scattering amplitude. The analysis can be extended to all orders of the low energy expansion. Clustering implies that the leading vertices also determine the leading contributions to the low energy singularities generated by multiparticle exchange. At next-to-leading order, only the cuts due to two-particle exchange contribute. Removing these, the one-particle irreducible parts are free of singularities up to and including $O(p^4)$. The contributions of order p^4 may again be represented by corresponding vertices in the effective Lagrangian, etc.

In order to extend the effective theory from the scattering amplitude to the Green functions of QCD, one needs to analyze the low energy expansion of the matrix elements of the currents with the Goldstone bosons. The singularities occurring therein can be sorted out in the same manner as for the scattering amplitude. The net result is that it suffices to equip the effective Lagrangian with suitable vertices, which in addition to the field $\pi^i(x)$ and its derivatives also involve the external fields $v_\mu(x)$, $a_\mu(x)$, $m(x)$, $\theta(x)$ and their derivatives. This then establishes claim (a).

Up to here, the analysis is perfectly general and applies to any system with a spontaneously broken Lie group. If \mathcal{L}_{eff} is invariant under a local group

of symmetries, then the same holds for the effective action that it generates – dimensional regularization manifestly preserves the gauge invariance of a scalar field theory, so that the symmetries of the classical Lagrangian also represent symmetries of the path integral. Claim (b) states that, for Lorentz invariant theories in four dimensions such as QCD, the converse is also true: The symmetries of the effective action imply that the effective Lagrangian can be brought to invariant form, except for the WZW term. The statement does not hold in the general case. In three dimensions, for instance, Chern-Simons theory represents a counter example. In four dimensions, the effective Lagrangian relevant for the spin waves of a ferromagnet does not have the same symmetries as the corresponding effective action: Under local spin rotations, the relevant Lagrangian changes by a total derivative.²⁹

The main problem encountered in the analysis of the symmetry properties of the effective theory is that the dynamical variables $\pi^i(x)$ do not have immediate physical significance. They merely serve as variables of integration in the path integral – the underlying theory does not contain such quantities. One may, for instance, subject the variables to a transformation of the type $\pi^i(x)' = f^i(\pi)$ without changing the content of the effective theory. The freedom corresponds to the fact that the off-shell extrapolation of matrix elements such as $\langle 0 | A_i^\mu | \pi^k \rangle$ or of the scattering amplitude is arbitrary. The explicit form of \mathcal{L}_{eff} , however, does depend on the choice of the variables. This implies that the effective Lagrangian is not unique – a circumstance that makes it rather tedious to establish its properties. In practical applications, the problem manifests itself through the freedom of adding terms of nonleading order to the effective Lagrangian that are proportional to the classical equation of motion. Since such terms can be removed with a suitable change of variables, they are irrelevant. The various contributions generated by two Lagrangians that only differ in this manner, however, are not the same – only the sum relevant for physical quantities is.

At leading order of the low energy expansion, the effective action is given by the tree graphs of $\mathcal{L}^{(2)}$, that is by the extremum of the corresponding classical action. The tensor $g_{ik}(\pi)$, that collects the effective coupling constants associated with the leading order Goldstone boson interaction vertices, plays the role of a metric on the space of the effective fields. The invariance of the effective action implies that this metric admits a group of isometries. The relevant Killing vectors also show up in the effective Lagrangian: They represent the coefficients of the terms that are linear in the external vector and axial fields. The effective fields π^i may be viewed as the coordinates of the quotient space G/H , where G and H are the symmetry groups of the Hamiltonian and of the vacuum, respectively. This space carries an intrinsic metric: The one

induced by the metric on the Lie group G . In the case of QCD, where $G/H = \text{SU}(3)$, the metric relevant for the leading term in the derivative expansion of the effective Lagrangian differs from the intrinsic metric of the group $\text{SU}(3)$ only by an overall factor, given by the square of the pion decay constant:

$$ds^2 = \sum_{i,k} g_{ik}(\pi) d\pi^i d\pi^k = \frac{1}{4} F^2 \langle dU dU^\dagger \rangle .$$

This shows that the low energy structure of the theory is determined by group geometry and explains why a model of mathematical physics turns out to be of relevance for our understanding of nature: For symmetry reasons, the leading order effective Lagrangian of QCD is the one of the nonlinear σ -model.

The analysis extends to all orders of the derivative expansion:¹⁹ With a suitable choice of the variables, the effective Lagrangian is invariant under local chiral rotations, except for the contributions generated by the anomalies.

12 Effective Lagrangian of next-to-leading order

By construction, the effective theory yields the general solution of the Ward identities obeyed by the Green functions of QCD. At leading order of the low energy expansion, this solution is fully determined by the three constants F , B and H_0 . At next-to-leading order, the general solution of the Ward identities contains 12 additional parameters, even if the singlet vector and axial currents and the winding number density are omitted. Their inclusion requires 11 further effective coupling constants – the explicit expression for $\mathcal{L}^{(4)}$ contains a plethora of terms.

Lorentz invariance implies that the Green functions can be decomposed into scalar functions, with coefficients that contain the external momenta and the tensors $g_{\mu\nu}$, $\epsilon_{\mu\nu\rho\sigma}$. In view of the fact that the square of $\epsilon_{\mu\nu\rho\sigma}$ can be expressed in terms of $g_{\mu\nu}$, there are two categories of contributions: The natural parity part of the effective action, which collects the pieces that do not contain the ϵ -tensor, and the unnatural parity part, where this tensor occurs exactly once. Since the contributions from the anomalies are proportional to the tensor $\epsilon_{\mu\nu\rho\sigma}$, they only affect the unnatural parity part of the effective action. For the natural parity part, chiral symmetry thus amounts to a very simple statement: This part of the effective action is invariant under local chiral rotations of the external fields.

It is convenient to decompose the effective Lagrangian accordingly. The natural parity part, in particular, the leading term of the derivative expansion, is invariant under local chiral rotations. Most of the terms occurring at

first nonleading order also belong to the natural parity part of the effective Lagrangian. The full expression for this part reads:²⁵

$$\begin{aligned}
\mathcal{L}_{np}^{(4)} = & L_1 \langle D_\mu U^\dagger D^\mu U \rangle^2 + L_2 \langle D_\mu U^\dagger D_\nu U \rangle \langle D^\mu U^\dagger D^\nu U \rangle \\
& + L_3 \langle D_\mu U^\dagger D^\mu U D_\nu U^\dagger D^\nu U \rangle + L_4 \langle D_\mu U^\dagger D^\mu U \rangle \langle U^\dagger \chi + \chi^\dagger U \rangle \\
& + L_5 \langle D_\mu U^\dagger D^\mu U (U^\dagger \chi + \chi^\dagger U) \rangle + L_6 \langle U^\dagger \chi + \chi^\dagger U \rangle^2 \\
& + L_7 \langle U^\dagger \chi - \chi^\dagger U \rangle^2 + L_8 \langle U^\dagger \chi U^\dagger \chi + \chi^\dagger U \chi^\dagger U \rangle \quad (40) \\
& - i L_9 \langle F_{\mu\nu}^R D^\mu U D^\nu U^\dagger + F_{\mu\nu}^L D^\mu U^\dagger D^\nu U \rangle + L_{10} \langle F_{\mu\nu}^R U F^{L\mu\nu} U^\dagger \rangle \\
& - i L_{11} D_\mu \theta \langle U^\dagger D^\mu U D_\nu U^\dagger D^\nu U \rangle + L_{12} D_\mu \theta D^\mu \theta \langle D_\nu U^\dagger D^\nu U \rangle \\
& + L_{13} D_\mu \theta D_\nu \theta \langle D^\mu U^\dagger D^\nu U \rangle + L_{14} D_\mu \theta D^\mu \theta \langle U^\dagger \chi + \chi^\dagger U \rangle \\
& - i L_{15} D_\mu \theta \langle D^\mu U^\dagger \chi - D^\mu U \chi^\dagger \rangle + i L_{16} \partial_\mu D^\mu \theta \langle U^\dagger \chi - \chi^\dagger U \rangle \\
& + H_1 \langle F_{\mu\nu}^R F^{R\mu\nu} + F_{\mu\nu}^L F^{L\mu\nu} \rangle + H_2 \langle \chi^\dagger \chi \rangle \\
& + H_3 v_{\mu\nu}^0 v^{0\mu\nu} + H_4 a_{\mu\nu}^0 a^{0\mu\nu} + H_5 (D_\mu \theta D^\mu \theta)^2 + H_6 (\partial_\mu D^\mu \theta)^2 ,
\end{aligned}$$

with $\chi(x) \equiv 2Bm(x)$. Some of the couplings involve the field strengths of the external fields: The traceless matrices $F_{\mu\nu}^R, F_{\mu\nu}^L$ collect the octet components of the field strengths belonging to the right- and left-handed fields $r_\mu = v_\mu + a_\mu$, $l_\mu = v_\mu - a_\mu$, respectively, while $v_{\mu\nu}^0, a_{\mu\nu}^0$ are the abelian field strengths of the singlets v_μ^0, a_μ^0 .

The main term in the unnatural parity part is the Wess-Zumino-Witten Lagrangian, which does not involve unknown coupling constants, but there is one extra term, relevant for Green functions involving the operators A_0^μ or ω :

$$\mathcal{L}_{up}^{(4)} = \mathcal{L}_{\text{wzw}} + i L_{17} \epsilon^{\mu\nu\rho\sigma} D_\mu \theta \langle F_{\nu\rho}^R D_\sigma U U^\dagger - F_{\nu\rho}^L U^\dagger D_\sigma U \rangle . \quad (41)$$

A detailed discussion of the structure of the Wess-Zumino-term in the presence of the external singlet fields v_μ^0, a_μ^0 and θ is given in the paper quoted above.²⁵

The coupling constants L_1, L_2 and L_3 multiply vertices that contain four or more Goldstone bosons; these couplings, for instance, occur at first nonleading order in the low energy representation of the scattering amplitude. L_4 and L_5 determine the expansion of the decay constants to first order in the quark masses and L_6, L_7, L_8 are relevant for the corresponding expansion of M_π, M_K, M_η . The coupling constant L_9 enters, for example, in the low energy expansion of the vector form factor, while L_{10} is relevant for the decay $\pi \rightarrow e\nu\gamma$. As the constants L_{11}, \dots, L_{17} multiply vertices that disappear if the external singlet fields θ and $\langle a_\mu \rangle$ are switched off, they only matter for Green functions formed with the singlet operators A_0^μ, ω , such as the topological susceptibility.

The couplings H_1, \dots, H_6 represent contact terms. Some of these are subject to a renormalization problem similar to the one occurring in H_0 . The

QCD Lagrangian, for instance, contains a term of the form $h_2 \langle m^\dagger m \rangle$, which is needed to absorb the quadratic divergences occurring in the graphs for the correlation functions of the scalar and pseudoscalar densities. As a consequence, the matrix elements $\langle 0 | \bar{u}u | 0 \rangle$, $\langle 0 | \bar{d}d | 0 \rangle$ and $\langle 0 | \bar{s}s | 0 \rangle$ contain a term linear in the quark masses that is inherently ambiguous. Within the effective theory, the problem concerns the value of H_2 and shows up already at leading order in the $1/N_c$ expansion. In principle, it should be possible to fix the ambiguity with the behaviour of the quark condensate when the quark masses become large: On physical grounds, the condensate should tend to zero – this can be the case only for one particular choice of the constant h_2 , so that the value of H_2 is fixed by this condition.

13 Perturbation theory

The path integral of the effective theory may be evaluated perturbatively. The leading term of the perturbation series is given by the tree graphs of $\mathcal{L}^{(2)}$. At first nonleading order, both the tree graphs of $\mathcal{L}^{(4)}$ and the one-loop graphs generated by $\mathcal{L}^{(2)}$ contribute. At next-to-next-to leading order, the two-loop graphs of $\mathcal{L}^{(2)}$ and the one-loop graphs containing one vertex from $\mathcal{L}^{(4)}$ need also be taken into account, together with the tree graphs of $\mathcal{L}^{(6)}$, etc.

The perturbative evaluation of the path integral is based on the decomposition $\mathcal{L}_{eff} = \mathcal{L}_{kin} + \mathcal{L}_{int}$, where the kinetic part is quadratic in the fields $\pi^i(x)$. Most of the preceding discussion concerns the properties of QCD when the quark masses are turned off, where $\mathcal{L}_{kin} = \frac{1}{2} \partial\pi \partial\pi$, so that the perturbation series involves massless scalar propagators. The effects generated by the quark masses are, however, accounted for in the effective Lagrangian – the perturbation series may just as well be worked out for nonzero quark masses. The position of the poles contained in the tree graphs of $\mathcal{L}^{(2)}$ are determined by the contributions quadratic in $\pi^i(x)$. Ignoring the isospin breaking due to the difference between m_u and m_d , the eigenvalues are given by

$$\overset{\circ}{M}_\pi^2 = 2\hat{m}B \quad , \quad \overset{\circ}{M}_K^2 = (\hat{m} + m_s)B \quad , \quad \overset{\circ}{M}_\eta^2 = \frac{2}{3}(\hat{m} + 2m_s)B \quad , \quad (42)$$

with $\hat{m} = \frac{1}{2}(m_u + m_d)$. The tree graphs of $\mathcal{L}^{(4)}$ and the one-loop graphs of $\mathcal{L}^{(2)}$ generate corrections of order m^2 . At first nonleading order, the result for the pion mass, for instance reads

$$\begin{aligned} M_\pi^2 = \overset{\circ}{M}_\pi^2 \left\{ 1 - \frac{16\hat{m}B}{F^2}(L_5 - 2L_8) - \frac{16(2\hat{m} + m_s)B}{F^2}(L_4 - 2L_6) \right. \\ \left. + \frac{1}{2F^2} \Delta(0, \overset{\circ}{M}_\pi) + \frac{1}{6F^2} \Delta(0, \overset{\circ}{M}_\eta) \right\} + O(m^3) \quad . \end{aligned} \quad (43)$$

The term $\Delta(0, M)$ stands for the scalar propagator at the origin,

$$\Delta(0, M) = \frac{1}{(2\pi)^4} \int \frac{dk}{M^2 - k^2 - i\epsilon} . \quad (44)$$

It arises from a tadpole graph that describes the propagation of a pseudoscalar particle of mass M , emitted and absorbed at one and the same point of space-time. The integral diverges quadratically. In dimensional regularization, it contains a pole at $d = 4$:

$$\begin{aligned} \Delta(0, M) &= 2M^2\lambda + \frac{1}{16\pi^2} M^2 \ln \frac{M^2}{\mu^2} , \\ \lambda &= \frac{\mu^{d-4}}{(4\pi)^2} \left\{ \frac{1}{d-4} - \frac{1}{2} (\ln 4\pi + \Gamma'(1) + 1) \right\} . \end{aligned} \quad (45)$$

The divergence can be absorbed in a renormalization of the effective coupling constants:

$$L_n = L_n^r + \Gamma_n \lambda . \quad (46)$$

The pion mass stays finite when $d \rightarrow 4$, provided the coefficients Γ_n obey $\Gamma_5 - 2\Gamma_8 = \frac{1}{6}$, $\Gamma_4 - 2\Gamma_6 = -\frac{1}{36}$. Actually, the renormalization coefficients of all of the coupling constants occurring at first nonleading order are known. Those for $L_1 \dots, L_{10}, H_1, H_2$ were worked out long ago.¹⁴ The additional couplings $L_{11} \dots, L_{17}, H_3, \dots, H_6$, which are needed to analyze the Green functions of the singlet currents and of the winding number density, do not pick up renormalization – the coefficients $\Gamma_{11}, \Gamma_{12}, \dots$ all vanish.²⁵

The formula (43) shows that the expansion of M_π^2 in powers of the quark masses does not represent a straightforward Taylor series, but contains logarithmic terms of the type $m^2 \ln m$. These so-called chiral logarithms are characteristic of chiral perturbation theory and occur in many of the results. Their coefficient is determined by chiral symmetry, in terms of the pion decay constant. In the above expressions, the scale of the logarithms is the running scale μ of dimensional regularization. The renormalized coupling constants L_n^r also depend on this scale. In the results of physical interest, the running scale drops out. If we wish, we may write the above formula for M_π in the form

$$M_\pi^2 = \overset{\circ}{M}_\pi^2 \left\{ 1 + \frac{\overset{\circ}{M}_\pi^2}{32\pi^2 F^2} \ln \frac{\overset{\circ}{M}_\pi^2}{\Lambda_A^2} - \frac{\overset{\circ}{M}_\eta^2}{96\pi^2 F^2} \ln \frac{\overset{\circ}{M}_\eta^2}{\Lambda_B^2} \right\} + O(m^3) . \quad (47)$$

The two scales Λ_A, Λ_B are independent of μ – their values are determined by the coupling constants L_4, L_5, L_6, L_8 .

14 Illustration: Form factors

As a further illustration, I consider the electromagnetic form factor of the pion,

$$\langle \pi^+(p') | j^\mu | \pi^+(p) \rangle = (p^\mu + p'^\mu) f_{\pi^+}(t) . \quad (48)$$

In this case, perturbation theory leads to the following representation:

$$f_{\pi^+}(t) = 1 + \frac{t}{F^2} \{ 2 L_9 + 2\phi(t, M_\pi) + \phi(t, M_K) \} + O(p^4) . \quad (49)$$

The leading term of the expansion is trivial – it represents the charge of the particle, $f_{\pi^+}(0) = 1$. At first nonleading order, there are two types of contributions: (i) The term proportional to L_9 , which comes from a tree graph containing a vertex from $\mathcal{L}^{(4)}$; it is linear in the momentum transfer t . (ii) The functions $\phi(t, M_\pi)$ and $\phi(t, M_K)$ are generated by one-loop graphs, which exclusively involve vertices from $\mathcal{L}^{(2)}$; they are nontrivial functions of t , containing branch cuts that start at $t = 4M_\pi^2$ and $t = 4M_K^2$. In dispersive language, the cuts are generated by $\pi\pi$ and $K\bar{K}$ intermediate states.

The function $\phi(t, M)$ may be expressed in terms of the scalar loop integral formed with two propagators:

$$J(p^2, M) = \frac{1}{(2\pi)^4} \int \frac{dk}{(M^2 - k^2 - i\epsilon)(M^2 - (k - p)^2 - i\epsilon)} , \quad (50)$$

which is logarithmically divergent, so that the divergent part is momentum independent: The difference $\bar{J}(t, M) \equiv J(t, M) - J(0, M)$ approaches a finite limit when $d \rightarrow 4$. The explicit expression reads:

$$\begin{aligned} J(t, M) &= \bar{J}(t, M) - 2\lambda - \frac{1}{16\pi^2} \left\{ \ln \frac{M^2}{\mu^2} + 1 \right\} , \\ \bar{J}(t, M) &= \frac{1}{16\pi^2} \left\{ \sigma \ln \frac{\sigma - 1}{\sigma + 1} + 2 \right\} , \quad \sigma = \left\{ 1 - \frac{4M^2}{t} \right\}^{\frac{1}{2}} . \end{aligned}$$

In this notation, the function $\phi(t, M)$ is given by

$$\phi(t, M) = \frac{1}{12} \left\{ (t - 4M^2) \bar{J}(t, M) - 2\lambda t - \frac{t}{16\pi^2} \left(\ln \frac{M^2}{\mu^2} + \frac{1}{3} \right) \right\} .$$

The resulting explicit representation of the form factor shows that the divergence of the one-loop contributions is absorbed by a renormalization of the coupling constant L_9 according to eq. (46), with $\Gamma_9 = \frac{1}{4}$.

The corresponding expression for the charge radius becomes

$$\langle r^2 \rangle_V^\pi = \frac{12L_9^r}{F^2} - \frac{1}{32\pi^2 F^2} \left\{ 2 \ln \frac{M_\pi^2}{\mu^2} + \ln \frac{M_K^2}{\mu^2} + 3 \right\} + O(m) . \quad (51)$$

The formula involves the coupling constant L_9 . Since the effective Lagrangian is consistent with chiral symmetry for any value of the coupling constants, symmetry alone does not determine the charge radius. It does, however, relate different observables. The slope of the K_{e3}^0 form factor $f_+(t)$, for instance, is also fixed by L_9 . Conversely, the experimental value of this slope,³⁰ $\lambda_+ = 0.0300 \pm 0.0016$, can be used to first determine the magnitude of L_9 and then to calculate the pion charge radius. This gives $\langle r^2 \rangle_V^\pi = 0.42 \text{ fm}^2$, to be compared with the experimental result obtained by scattering pions on atomic electrons:³¹ $0.439 \pm 0.008 \text{ fm}^2$.

In the case of the neutral kaon, the analogous representation reads

$$f_{K^0}(t) = \frac{t}{F^2} \{-\phi_\pi(t) + \phi_K(t)\} + O(t^2, tm). \quad (52)$$

A term of order one does not occur here, because the charge vanishes. There is no contribution from $\mathcal{L}^{(4)}$, either. Chiral perturbation theory thus provides a parameter free prediction in terms of the one-loop integrals $\phi_\pi(t), \phi_K(t)$. In particular, up to corrections of $O(m)$, the slope of the form factor is given by³²

$$\langle r^2 \rangle_V^{K^0} = -\frac{1}{16\pi^2 F^2} \ln \frac{M_K}{M_\pi} = -0.04 \text{ fm}^2 , \quad (53)$$

to be compared with the experimental value³³ $-0.054 \pm 0.026 \text{ fm}^2$. In the meantime, similar parameter free one-loop predictions have been discovered for quite a few other observables.¹⁵

The above expression for the charge radius exhibits another interesting feature, which is related to the fact that the cloud of Goldstone bosons that surrounds the pion becomes long range if the quark masses are sent to zero: The charge radius tends to infinity in that limit. The phenomenon also shows up in the behaviour of the form factor in the massless theory, where the expansion in powers of the momentum transfer contains a nonanalytic term,

$$f_{\pi^+}(t) = 1 - \frac{1}{64\pi^2 F^2} t \ln \frac{(-t)}{\Lambda_C^2} + O(t^2) , \quad (54)$$

so that the first derivative explodes at $t = 0$. The scale of the logarithm occurring here is fixed by the coupling constant L_9 :

$$L_9^r = \frac{1}{128\pi^2} \left\{ \ln \frac{\Lambda_C^2}{\mu^2} - \frac{5}{3} \right\} . \quad (55)$$

For the present review, the above sample calculations must suffice to illustrate the nature of the results obtained at first nonleading order of the chiral perturbation series. Plenty of such one-loop results are reported in the literature.¹⁵ In quite a few cases, the series has been worked out to next-to-next-to leading order, where the two-loop graphs give rise to double chiral logarithms.^{34–37} The explicit form of $\mathcal{L}^{(6)}$ is known,³⁸ as well as the renormalization of the effective coupling constants occurring therein.³⁹ The renormalization group flow of the effective theory was also examined, in particular in view of infrared attractive fixed points.⁴⁰

Among other things, the evaluation of the e.m. form factor to two loops allows a determination of the pion charge radius that is free of the model assumptions underlying the “experimental” value quoted above. The result of the model independent analysis reads:³⁵ $\langle r^2 \rangle_V^\pi = 0.437 \pm 0.016 \text{ fm}^2$.

15 Magnitude of the coupling constants

One of the main problems encountered in the effective Lagrangian approach is the occurrence of an entire fauna of effective coupling constants. If these constants are treated as totally arbitrary parameters, the predictive power of the method is nil — as a bare minimum, an estimate of their order of magnitude is needed.

In principle, the effective coupling constants F, B, L_1, L_2, \dots are calculable. They do not depend on the light quark masses, but are determined by the scale Λ_{QCD} and by the masses of the heavy quarks. The available, admittedly crude evaluations of F and B on the lattice demonstrate that the calculation is even feasible in practice. As discussed above, the coupling constants L_1, L_2, \dots are renormalized by the logarithmic divergences occurring in the one-loop graphs. This property sheds considerable light on the structure of the chiral expansion and provides a rough estimate for the order of magnitude of the effective coupling constants.⁴¹ The point is that the contributions generated by the loop graphs are smaller than the leading (tree graph) contribution only for momenta in the range $|p| \lesssim \Lambda_\chi$, where

$$\Lambda_\chi \equiv 4\pi F / \sqrt{N_f} \quad (56)$$

is the scale occurring in the coefficient of the logarithmic divergence (N_f is the number of light quark flavours). This indicates that the low energy expansion is an expansion in powers of $(p/\Lambda_\chi)^2$, with coefficients of order one. The argument also applies to the expansion in powers of m_u, m_d and m_s , indicating that the relevant expansion parameter is given by $(M_\pi/\Lambda_\chi)^2$ and $(M_K/\Lambda_\chi)^2$, respectively.

A more quantitative picture may be obtained along the following lines.¹³ Consider again the e.m. form factor of the pion and compare the chiral representation (49) with the dispersion relation

$$f_{\pi^+}(t) = \frac{1}{\pi} \int_{4M_\pi^2}^{\infty} \frac{dt'}{t' - t} \text{Im} f_{\pi^+}(t') .$$

In this relation, the contributions ϕ_π, ϕ_K from the one-loop graphs of chiral perturbation theory correspond to $\pi\pi$ and $K\bar{K}$ intermediate states. To leading order in the low energy expansion, the corresponding imaginary parts are slowly rising functions of t . The most prominent contribution on the right-hand side, however, stems from the region of the ρ -resonance, which nearly saturates the integral: The vector meson dominance formula, $f_{\pi^+}(t) = (1 - t/M_\rho^2)^{-1}$, which results if all other contributions are dropped, provides a perfectly decent representation of the form factor for small values of t . In particular, this formula predicts $\langle r^2 \rangle_V^\pi = 0.39 \text{ fm}^2$, in satisfactory agreement with observation. This implies that the effective coupling constant L_9 is approximately given by

$$L_9 = \frac{F^2}{2M_\rho^2} . \quad (57)$$

In the channel under consideration, the pole due to ρ exchange thus represents the dominating low energy singularity — the $\pi\pi$ and $K\bar{K}$ cuts merely generate a small correction. More generally, the validity of the vector meson dominance formula shows that, for the e.m. form factor, the scale of the low energy expansion is set by $M_\rho = 770 \text{ MeV}$.

Analogous estimates can be given for all effective coupling constants at order p^4 , saturating suitable dispersion relations with contributions from resonances,^{42,43} for instance:

$$L_5 = \frac{F^2}{4M_S^2} , \quad L_7 = -\frac{F^2}{48M_{\eta'}^2} ,$$

where $M_S \simeq 980 \text{ MeV}$ and $M_{\eta'} \simeq 958 \text{ MeV}$ are the masses of the scalar octet and pseudoscalar singlet, respectively. In all those cases, where direct phenomenological information is available, these estimates do remarkably well. I conclude that the observed low energy structure is dominated by the poles and cuts generated by the lightest particles — hardly a surprise.

The effective theory is constructed on the asymptotic states of QCD. In the sector with zero baryon number, charm, beauty, \dots , the Goldstone bosons form a complete set of such states, all other mesons being unstable against decay into these (strictly speaking, the η occurs among the asymptotic states

only for $m_d = m_u$; it must be included among the degrees of freedom of the effective theory, nevertheless, because the masses of the light quarks are treated as a perturbation — in massless QCD, the poles generated by the exchange of this particle occur at $p = 0$). The Goldstone degrees of freedom are explicitly accounted for in the effective theory — they represent the dynamical variables. All other levels manifest themselves only indirectly, through the values of the effective coupling constants. In particular, low lying states such as the ρ generate relatively small energy denominators, giving rise to relatively large contributions to some of these coupling constants.

In some channels, the scale of the chiral expansion is set by M_ρ , in others by the masses of the scalar or pseudoscalar states occurring around 1 GeV. This confirms the rough estimate (56). The cuts generated by Goldstone pairs are significant in some cases and are negligible in others, depending on the numerical value of the relevant Clebsch-Gordan coefficient. If this coefficient turns out to be large, the coupling constant in question is sensitive to the renormalization scale used in the loop graphs. The corresponding pole dominance formula is then somewhat fuzzy, because the prediction depends on how the resonance is split from the continuum underneath it.

More precise results can be obtained by evaluating suitable dispersion integrals or sum rules, using unitarity to determine the relevant imaginary parts.^{44,45} This method makes it evident that the pole dominance formulae only represent a crude parametrization: The value of M_S , for instance, is the scale at which the relevant integral over the imaginary part receives its main contributions — it is inessential whether or not that contribution is adequately described by a narrow peak.⁴⁶

The quantitative estimates of the effective couplings given above explain why it is justified to treat m_s as a perturbation. At order p^4 , the symmetry breaking part of the effective Lagrangian is determined by the constants L_4, \dots, L_8 . These constants are immune to the low energy singularities generated by spin 1 resonances, but are affected by the exchange of scalar or pseudoscalar particles, so that their magnitude is determined by the scale $M_S \simeq M_{\eta'} \simeq 1$ GeV. Accordingly, the expansion in powers of m_s is controlled by the parameter $(M_K/M_S)^2 \simeq \frac{1}{4}$. The asymmetry in the decay constants, for example, is determined by L_5 . The estimate of this coupling constant given above yields

$$\frac{F_K}{F_\pi} \simeq 1 + \frac{M_K^2 - M_\pi^2}{M_S^2} ,$$

up to chiral logarithms and higher order terms. This shows that the breaking of the chiral and eightfold way symmetries is controlled by the mass ratio of the Goldstone bosons to the non-Goldstone states of spin zero. In chiral per-

turbation theory, the observation that the Goldstones are the lightest hadrons thus acquires quantitative significance: For momentum independent quantities such as masses, decay constants, charge radii or scattering lengths, the magnitude of consecutive orders in the chiral perturbation series is determined by the ratio $(M_K/M_S)^2$.

16 Partition function

Chiral perturbation theory yields remarkable insights into the equilibrium properties of the theory at temperatures below the chiral phase transition. The same effective Lagrangian that provides a representation for the Green functions of QCD also yields a representation for the partition function,⁴⁷

$$\text{Tr} \exp \left(-\frac{H_{\text{QCD}}}{kT} \right) = \mathcal{N}_{\text{eff}}^E \int [d\pi] \exp \left(-\int d^3x \int_0^\beta dx^4 \mathcal{L}_{\text{eff}}^E \right) , \quad (58)$$

where $\mathcal{L}_{\text{eff}}^E$ is the euclidean form of the effective Lagrangian, and the path integral extends over all configurations that are periodic in the time direction, $\pi^i(\vec{x}, x^4 + \beta) = \pi^i(\vec{x}, x^4)$, with $\beta = 1/kT$. In particular, the melting of the quark condensate, which sets in when the temperature rises, can be worked out by means of this formula. If the masses m_u and m_d are set equal to zero, the temperature expansion takes the form,⁴⁸

$$\langle \bar{u}u \rangle_T = \langle 0 | \bar{u}u | 0 \rangle \left\{ 1 - \frac{T^2}{8\bar{F}^2} - \frac{T^4}{384\bar{F}^4} - \frac{T^6}{288\bar{F}^6} \ln \left(\frac{T_1}{T} \right) + O(T^8) \right\} ,$$

where \bar{F} is the value of F_π in the limit $m_u = m_d = 0$. The formula is exact – for massless quarks, the temperature scale relevant at low T is the pion decay constant. The additional logarithmic scale T_1 occurring at order T^6 is determined by the effective coupling constants that enter the expression for the effective Lagrangian at order p^4 . Since these are known from the phenomenology of $\pi\pi$ scattering, the numerical value of T_1 is also known:⁴⁸ $T_1 = 470 \pm 110$ MeV.

While the Goldstone bosons give rise to powers of the temperature, massive states like the ρ -meson are suppressed by Boltzmann factors like $\exp(-M_\rho/kT)$. In view of the fact that the spectrum contains many such states, these nevertheless generate a significant contribution, already for temperatures of order 140 MeV (there, the typical energy of the Goldstone bosons is of order $2.7 kT \simeq 400$ MeV). The massive states accelerate the melting process.

The low temperature expansion clearly exhibits the limitations of the method: The truncated series can be trusted only at low temperatures, where

the first term represents the dominant contribution. In particular, the behaviour of the quark condensate in the vicinity of the chiral phase transition is beyond the reach of the effective theory discussed here.

The low temperature expansion was investigated for quite a few other quantities of physical interest. The position and the residue of the pole in the thermal correlation function of the axial current (effective values of M_π and F_π) are known up to and including two loops.⁴⁹ For massive particles, the sensitivity of the mass and of the width to the temperature was also analyzed in detail.⁵⁰ These results are of interest, in particular, for the physics of the final state produced in heavy ion collisions. Some of the corresponding transport coefficients have been worked out by means of chiral perturbation theory.⁵¹ The effects due to an external magnetic field⁵² have also been investigated. Recently, the behaviour of QCD at high chemical potential, in particular, the occurrence of a “colour–flavour locking phase” has attracted great interest.⁵³ The properties of such a phase can also be analyzed by means of a suitable effective Lagrangian.⁵⁴

17 Universality

The Higgs sector of the Standard Model is also characterized by a spontaneously broken symmetry. In that case, the Hamiltonian is symmetric under $G = O(4)$, while the symmetry group of the ground state is the subgroup of those rotations that leave the expectation value $\langle 0 | \vec{\varphi} | 0 \rangle$ of the Higgs field invariant, $H = O(3)$. As discussed in section 11, the structure of the effective Lagrangian follows from the Ward identities obeyed by the Green functions. The form of these identities is controlled by the structure of G and H in the infinitesimal neighbourhood of the neutral element. Since the groups $O(4)$ and $O(3)$ are locally isomorphic to $SU(2)_R \times SU(2)_L$ and $SU(2)_V$, respectively, the effective Lagrangians relevant for the Higgs model and for QCD with two massless flavours are identical. At the level of the effective theory, the only difference between these two physically quite distinct systems is that the numerical values of the effective coupling constants are different. In the case of QCD, the one occurring at leading order of the derivative expansion is the pion decay constant, $\bar{F} \simeq 90 \text{ MeV}$, while in the Higgs model, this coupling constant is larger by more than three orders of magnitude, $\bar{F} \simeq 250 \text{ GeV}$. At next-to-leading order, the effective coupling constants are also different. In particular, in QCD, the anomaly coefficient is equal to N_c , while in the Higgs model, it vanishes.

The operators relevant for the expectation values $\langle 0 | \bar{q}q | 0 \rangle$, $\langle 0 | \vec{\varphi} | 0 \rangle$ transform in the same manner under G . The above formula for the quark condensate

thus holds, without any change whatsoever, also for the Higgs condensate:

$$\langle \vec{\varphi} \rangle_T = \langle 0 | \vec{\varphi} | 0 \rangle \left\{ 1 - \frac{T^2}{8\bar{F}^2} - \frac{T^4}{384\bar{F}^4} - \frac{T^6}{288\bar{F}^6} \ln\left(\frac{T_1}{T}\right) + O(T^8) \right\} .$$

In fact, the universal term of order T^2 was discovered in the framework of the Higgs model, in connection with work on the electroweak phase transition.⁵⁵

The example illustrates the physical nature of effective theories: At long wavelength, the microscopic structure does not play any role. The behaviour only depends on those degrees of freedom that require little excitation energy. The hidden symmetry, which is responsible for the absence of an energy gap and for the occurrence of Goldstone bosons, at the same time also determines their low energy properties. For this reason, the form of the effective Lagrangian is controlled by the symmetries of the system and is, therefore, universal. The microscopic structure of the underlying theory exclusively manifests itself in the numerical values of the effective coupling constants.

Chiral perturbation theory may also be used to analyze the spontaneous breakdown of electroweak gauge symmetry, without relying on the assumption that the phenomenon is described by the Higgs model. As far as the low energy structure is concerned, alternative models such as technicolour are described by the same effective theory – the degrees of freedom involved in the formation of the electroweak condensate only show up indirectly, in the values of the effective coupling constants.^{56,57}

18 Concluding remarks

The main motivation for working in chiral dynamics is that it is fun, but for those strolling in other fields, one can give a few scientific reasons that indicate what this is good for.

In condensed matter physics, effective theories have successfully been used since a long time. In particular, the phenomenon of spontaneous symmetry breakdown was discovered there. In this perspective, chiral perturbation theory is just another illustration of the fact that the relevant degrees of freedom must be identified to arrive at a good understanding of the physics. Lorentz invariance implies that the relation between the wavelength and the frequency of the Goldstone bosons of particle physics is given by $\omega = c|\vec{k}|$. The dispersion law for the magnons of an antiferromagnet is of the same form, but there are two differences: The value of c is not the same and, more importantly, the relation is linear in $|\vec{k}|$ only for large wavelengths – the dispersion law also contains higher powers of \vec{k} , while Lorentz invariance excludes such terms. The effective theories describing the magnons of a ferromagnet or the phonons of

a solid differ even more from those relevant for particle physics – it is very instructive to see why that is so.²⁹

Chiral perturbation theory has become an indispensable tool in the phenomenological analysis, because it provides a detailed understanding of the low energy properties of the strong interactions and puts early attempts in this direction – the static model of the πN interaction, just to name one – on firm mathematical grounds. In particular, much of what we know about the weak interactions comes from kaon decays. To analyze the relevant observables, the effects due to the strong interactions must be accounted for.

Also, what we know about the pattern of light quark masses heavily relies on the results obtained with chiral perturbation theory. Gradually, lattice calculations start contributing to our knowledge in this domain, but further progress with light dynamical fermions is required before the numbers obtained with this technique can be taken at face value, particularly for m_u and m_d – it is difficult to numerically simulate the effects generated by the emission and absorption of the Goldstone bosons. Even so, these calculations already now shed a considerable amount of light on the low energy properties of QCD.

In connection with lattice simulations, chiral perturbation theory is useful as a tool to analyze the finite size effects. Since these are dominated by the lightest particles, that is by the Goldstone bosons, they can be calculated on the basis of the effective theory. For quite a few observables, the calculation has been done.⁵⁸ Once lattice evaluations with dynamical quarks reach realistically small quark masses, these results should turn out to be very useful, because they allow one to correct the numerical results for the most important finite size effects, so that volumes of modest size should suffice – in the standard approach, where the infinite volume limit is performed by brute force, very large volumes are required.⁵⁹

The volume-dependence of the partition function is relevant also for an understanding of the spectrum of the Dirac operator. As pointed out by Banks and Casher,⁶⁰ the quark condensate is determined by the spectral density at small eigenvalues. Chiral perturbation theory allows one to establish a rather detailed picture for the properties of the spectrum, as well as for the distribution of the winding number.⁶¹

A further issue, where chiral perturbation theory turned out to be very useful, is the Okubo-Iizuka-Zweig rule, which becomes exact only in the limit where the number of colours is sent to infinity. The effective theory can be extended to cover the case where N_c is taken large. In this framework, the η' plays a crucial role, because the U(1)-anomaly, which is responsible for the bulk of the mass of this particle, is then suppressed.⁶² The systematic expansion of the effective theory in powers of $1/N_c$ allows one to disentangle the effects

that break the OZI rule from those that break flavour symmetry – a necessary prerequisite to quantitatively describe, for instance, $\eta - \eta'$ mixing.⁶³ The extended effective theory also provides a handle on the ambiguity pointed out by Kaplan and Manohar,⁶⁴ which is of relevance in connection with phenomenological determinations of the quark mass ratios: The ambiguity is suppressed to all orders of the $1/N_c$ expansion.²⁵

One of the problems encountered in chiral perturbation theory is that the truncated low energy expansion of the amplitudes satisfies unitarity only up to higher order contributions. In some cases, the effects generated by the two-particle cuts are quite large, already at threshold – in a one-loop calculation, these are accounted for only to leading order of the low energy expansion. In the case of the $\pi\pi$ scattering amplitude, a much more accurate representation is obtained by matching the chiral representation with a dispersive one, based on the Roy equations.⁶⁵ Dispersive methods have also been used to analyze the two-particle cuts in other amplitudes, such as two-point functions or form factors.^{44,45,66} An improved description may be obtained by rewriting the chiral representation in such a manner that elastic unitarity is obeyed exactly. In the case of a form factor, for instance, it suffices to replace the one-loop formula for $f(t)$ by the one for $1/f(t)$. The Gounaris-Sakurai formula⁶⁷ amounts to precisely this prescription – it provides a remarkably accurate low energy representation for the e.m. form factor. As advocated by Truong,⁶⁸ a similar unitarization procedure can be applied to other amplitudes as well. The main problem with this approach is that the choice of the unitarization is not unique. It would be of considerable interest to find out what reordering of the chiral perturbation series is required to improve the representation in an algebraically controlled manner.

A fascinating aspect of chiral dynamics is the possibility of subjecting the theory to experimental test. As an example, I briefly discuss the S -wave scattering lengths of $\pi\pi$ scattering. Since Goldstone bosons of zero momentum cannot interact, these scattering lengths vanish if m_u and m_d are sent to zero – like the pion mass, they represent a quantitative measure for the breaking of chiral symmetry due to the quark masses. In fact, in 1966, Weinberg showed that the two symmetry breaking effects are related to one another. In particular, the leading order prediction for the isoscalar scattering length reads:⁹

$$a_0^0 = \frac{7M_\pi^2}{32\pi F_\pi^2} + O(m^2) \quad .$$

The next-to-leading order terms in the low energy expansion of the $\pi\pi$ scattering amplitude were worked out¹³ in 1983 and, in 1996, those of next-to-next-to leading order were also calculated.³⁶ Matching the two-loop representation of

the scattering amplitude with the dispersive representation obtained by solving the Roy equations,^{65,69} the scattering lengths can be predicted at the 2–3% level of accuracy:^{70,71} $a_0^0 = 0.220 \pm 0.005$, to be compared with Weinberg’s leading order formula, which yields $a_0^0 = 0.16$ and with the one-loop result, $a_0^0 = 0.20$. In this case, the convergence of the expansion in powers of the quark masses is well understood – the error bar attached to the above numerical result includes the uncertainties due to terms of yet higher order.

The example represents one of the very rare cases in strong interaction physics, where theory is ahead of experiment. The symmetry breaking effects due to m_u, m_d are hard to measure, because they are very small. Currently, the best source of information concerning the value of the $\pi\pi$ scattering lengths is the decay $K \rightarrow \pi\pi e\nu$. The preliminary results of a recent measurement of this decay⁷² are consistent with the above prediction, but the experimental errors still leave room for significant deviations. There is a beautiful proposal to due to Nemenov,⁷³ based on the observation that $\pi^+\pi^-$ atoms decay into a pair of neutral pions, through the strong transition $\pi^+\pi^- \rightarrow \pi^0\pi^0$. Since the momentum transfer nearly vanishes, the decay rate is proportional to the square of the combination $a_0^0 - a_0^2$ of S -wave $\pi\pi$ scattering lengths. The properties of the pionic atom have recently been analyzed at depth, on the basis of chiral perturbation theory,⁷⁴ so that the precise form of the relation between the decay rate and the scattering lengths is known. Since the predictions for the latter are very sharp, the measurement of the lifetime of $\pi^+\pi^-$ atoms, which aims at an accuracy of 10%, will provide a very stringent test of the standard framework that I have been relying on throughout this review. This framework is based on the hypothesis that the Gell-Mann-Oakes-Renner relation is dominated by the order parameter of lowest dimension – the quark condensate. As emphasized by J. Stern and collaborators,^{75,45} symmetry alone does not guarantee that this is so. If the outcome of the experiment should turn out to be in conflict with the prediction quoted above, my understanding of chiral dynamics would undergo a first order phase transition.

Acknowledgement

I thank Roland Kaiser for useful comments and the Swiss National Science Foundation for support.

References

1. M. Goldberger and S. Treiman, *Phys. Rev.* **110** (1958) 1178.

2. For a detailed discussion, see Proc. MENU99, Zuoz, Switzerland (1999), *πN Newslett.* **15** (1999).
3. Y. Nambu, *Phys. Rev. Lett.* **4** (1960) 380.
4. J. Goldstone, *Nuovo Cim.* **19** (1961) 154.
5. W. Heisenberg, *Z. Phys.* **77** (1932) 1.
6. M. Gell-Mann and Y. Ne'eman, *The Eightfold Way*, W. A. Benjamin (New York, 1964);
M. Gell-Mann, *Physics* **1** (1964) 63.
7. S. L. Adler, *Phys. Rev.* **140** (1965) B736.
W. I. Weisberger, *Phys. Rev.* **143** (1966) 1302.
8. S. Weinberg, *Phys. Rev. Lett.* **16** (1966) 879.
9. S. Weinberg, *Phys. Rev. Lett.* **17** (1966) 616.
10. S. Coleman, J. Wess and B. Zumino, *Phys. Rev.* **177** (1969) 2239;
C. Callan, S. Coleman, J. Wess and B. Zumino, *Phys. Rev.* **177** (1969) 2247.
11. R. Dashen and M. Weinstein, *Phys. Rev.* **183** (1969) 1291;
L.-F. Li and H. Pagels, *Phys. Rev. Lett.* **26** (1971) 1204.
12. S. Weinberg, *Physica A* **96** (1979) 327.
13. J. Gasser and H. Leutwyler, *Phys. Lett. B* **125** (1983) 321;
Ann. Phys. (N.Y.) **158** (1984) 142.
14. J. Gasser and H. Leutwyler, *Nucl. Phys. B* **250** (1985) 465.
15. G. Ecker, in *Quantitative Particle Physics*, Cargèse 1992, eds. M. Lévy *et al.* (Plenum, New York, 1993) and in *Broken Symmetries*, Schladming, Austria (1998), hep-ph/9805500;
U. Meissner, *Rep. Prog. Phys.* **56** (1993) 903;
A. Dobado, A. Gomez-Nicola, J. P. Maroto and J. P. Pelaez, *Effective Lagrangians for the Standard Model*, Springer-Verlag, N.Y. 1997;
A. Pich, in *Probing the Standard Model of Particle Interactions*, Les Houches, France (1997), hep-ph/9806303;
J. Gasser, *Nucl. Phys. Proc. Suppl.* **86** (2000) 257;
G. Colangelo, hep-ph/0001256;
Barry R. Holstein; hep-ph/0001281;
Jose Wudka, hep-ph/0002180.
16. C. Vafa and E. Witten, *Nucl. Phys. B* **234** (1984) 173.
17. M. Gell-Mann, R. J. Oakes and B. Renner, *Phys. Rev.* **175** (1968) 2195.
18. W. Heisenberg and H. Euler, *Z. Phys.* **98** (1936) 714.
19. H. Leutwyler, *Ann. Phys. (N.Y.)* **235** (1994) 165.
20. R. J. Crewther, *Phys. Lett. B* **70** (1977) 349.
21. P. Di Vecchia and G. Veneziano, *Nucl. Phys. B* **171** (1980) 253.
22. G. M. Shore and G. Veneziano, *Phys. Lett. B* **244** (1990) 75;

- S. Narison, G. M. Shore and G. Veneziano, *Nucl. Phys. B* **433** (1995) 209; *ibid.* B **546** (1999) 235.
23. B. L. Ioffe and A. G. Oganesian, *Phys. Rev. D* **57** (1998) 6590;
B. L. Ioffe, *Phys. Atom. Nucl.* **62** (1999) 2052;
Heavy Ion Phys. **9** (1999) 29;
B. L. Ioffe and A. V. Samsonov, hep-ph/9906285.
 24. H. Leutwyler, *Phys. Lett. B* **378** (1996) 313.
 25. R. Kaiser and H. Leutwyler, hep-ph/0007101.
 26. P. Minkowski, *Phys. Lett. B* **237** (1990) 531.
 27. For a recent paper that also discusses earlier work, see
Jochen Heitger *et al.* [ALPHA collaboration], hep-lat/0006026.
 28. J. Wess and B. Zumino, *Phys. Lett.* **B37** (1971) 95;
E. Witten, *Nucl. Phys.* **B223** (1983) 422.
 29. H. Leutwyler, *Phys. Rev. D* **49** (1994) 3033;
Helv. Phys. Acta **70** (1997) 275;
C. Hofmann, hep-ph/9706418; cond-mat/9805277;
Don H. Kim and Patrick A. Lee, *Ann. Phys.* **272** (1999) 130.
 30. C. Caso *et al.* [Particle Data Group], *Eur. Phys. J. C* **3** (1998) 1.
 31. S. R. Amendolia *et al.* [NA7 collaboration],
Nucl. Phys. B **277** (1986) 168.
 32. J. Gasser and H. Leutwyler, *Nucl. Phys. B* **250** (1985) 517.
 33. W. R. Molzon, *Phys. Rev. Lett.* **41** (1978) 1213.
 34. J. Gasser and U. Meissner, *Nucl. Phys. B* **357** (1991) 90;
G. Colangelo, M. Finkemeier and R. Urech, *Phys. Rev. D* **54** (1996) 4403;
P. Post and K. Schilcher, hep-ph/0007095.
 35. J. Bijnens, G. Colangelo and P. Talavera, *JHEP* **9805** (1998) 014.
 36. J. Bijnens, G. Colangelo, G. Ecker, J. Gasser and M. E. Sainio,
Phys. Lett. B **374** (1996) 210; *Nucl. Phys. B* **508** (1997) 263;
ibid. **B517** (1998) 639 (E).
 37. E. Golowich and J. Kambor, *Phys. Rev. Lett.* **79** (1997) 4092;
Phys. Lett. B **421** (1998) 319; *Phys. Rev. D* **58** (1998) 036004;
S. Durr and J. Kambor, *Phys. Rev. D* **61** (2000) 114025;
G. Amoros, J. Bijnens and P. Talavera, *Nucl. Phys. B* **568** (2000) 319.
 38. H. Fearing and S. Scherer, *Phys. Rev. D* **53** (1996) 315;
J. Bijnens, G. Colangelo and G. Ecker, *JHEP* **9902** (1999) 020.
 39. J. Bijnens, G. Colangelo and G. Ecker, *Phys. Lett. B* **441** (1998) 437;
Ann. Phys. (N.Y.) **280** (2000) 100.
 40. D. U. Jungnickel and C. Wetterich, *Eur. Phys. J. C* **2** (1998) 557;
M. Atance and B. Schrempp, hep-ph/9912335.
 41. H. Georgi and A. Manohar, *Nucl. Phys. B* **234** (1984) 189;

- M. Soldate and R. Sundrum, *Nucl. Phys. B* **340** (1990) 1;
R. S. Chivukula, M. J. Dugan and M. Golden,
Phys. Rev. D **47** (1993) 2930.
42. G. Ecker *et al.*, *Nucl. Phys. B* **321** (1989) 311;
Phys. Lett. B **223** (1989) 425.
43. H. Leutwyler, *Nucl. Phys. B* **337** (1990) 108 and in *Yukawa couplings and the origin of mass*, Proc. 2nd IFT Workshop, University of Florida, Gainesville, Feb. 1994, ed. P. Ramond (International Press, Cambridge MA, 1995).
44. J. F. Donoghue, J. Gasser and H. Leutwyler,
Nucl. Phys. B **343** (1990) 341.
45. B. Moussallam, *Eur. Phys. J. C* **14** (2000) 111; hep-ph/0005245;
S. Descotes, L. Girlanda and J. Stern, *JHEP* **0001** (2000) 041;
S. Descotes and J. Stern, *Phys. Rev. D* **62** (2000) 504011;
hep-ph/0007082.
46. X.-J. Wang and M.-L. Yan, hep-ph/0001150, hep-ph/0004157.
47. J. Gasser and H. Leutwyler, *Phys. Lett. B* **184** (1987) 83.
48. P. Gerber and H. Leutwyler, *Nucl. Phys. B* **321** (1989) 387;
T. Hatsuda, *Nucl. Phys. A* **544** (1992) 27;
M. C. Birse, *Acta Phys. Polon. B* **29** (1998) 2357;
N. O. Agasian, D. Ebert and E. M. Ilgenfritz,
Nucl. Phys. A **637** (1998) 135;
A. V. Smilga, *Nucl. Phys. A* **654** (1999) 136C;
J. R. Pelaez, *Phys. Rev. D* **59** (1999) 014002;
A. Dobado and J. R. Pelaez, *Phys. Rev. D* **59** (1999) 034004.
49. A. Schenk, *Nucl. Phys. B* **363** (1991) 97;
D. Toublan, *Phys. Rev. D* **56** (1997) 5629.
50. H. Leutwyler and A. Smilga, *Nucl. Phys. B* **342** (1990) 302;
V. L. Eletsky and B. L. Ioffe, *Phys. Rev. Lett.* **78** (1997) 1010;
Phys. Lett. B **401** (1997) 327;
V. L. Eletsky, B. L. Ioffe and J. I. Kapusta,
Eur. Phys. J. A **3** (1998) 381;
S. Mallik and K. Mukherjee, *Phys. Rev. D* **61** (2000) 116007;
hep-ph/9809231;
V. Sheel, H. Mishra and J. C. Parikh, *Phys. Rev. D* **59** (1999) 034501;
R. Escribano, F. S. Ling and M. H. Tytgat,
Phys. Rev. D **62** (2000) 056004.
51. J. L. Goity and H. Leutwyler, *Phys. Lett. B* **228** (1989) 517;
G. M. Welke, R. Venugopalan and M. Prakash,
Phys. Lett. B **245** (1990) 137;

- H. Bebie *et al.*, *Nucl. Phys. B* **378** (1992) 95;
J. M. Martinez Resco and M. A. Valle Basagoiti,
Phys. Rev. D **58** (1998) 097901;
A. Gomez Nicola and V. Galan-Gonzalez, *Phys. Lett. B* **449** (1999) 288.
52. N. O. Agasian and I. A. Shushpanov, *JETP Lett.* **70** (1999) 717;
Phys. Lett. B **472** (2000) 143;
N. O. Agasian, hep-ph/0005300.
53. For a review, see
F. Wilczek, hep-ph/9908480;
K. Rajagopal, hep-ph/9908360.
54. R. Casalbuoni and R. Gatto, *Phys. Lett. B* **464** (1999) 111;
ibid. **469** (1999) 213; hep-ph/9911223.
55. P. Binétruy and M. K. Gaillard, *Phys. Rev. D* **32** (1985) 931.
56. T. Appelquist and C. Bernard, *Phys. Rev. D* **22** (1980) 200;
A. C. Longhitano, *Phys. Rev. D* **22** (1980) 1166.
57. A. Nyffeler and A. Schenk, *Phys. Rev. D* **62** (2000) 036002;
A. Nyffeler, hep-ph/9912472.
58. J. Gasser and H. Leutwyler, *Phys. Lett. B* **188** (1987) 477;
P. Hasenfratz and H. Leutwyler, *Nucl. Phys. B* **343** (1990) 241;
F. C. Hansen and H. Leutwyler, *Nucl. Phys. B* **350** (1991) 201.
Chiral perturbation theory can also be applied to the quenched approximation on the lattice, see for instance:
C. Bernard and M. Golterman, *Phys. Rev. D* **46** (1992) 853;
S. R. Sharpe, *Phys. Rev. D* **46** (1992) 3146; *Phys. Rev. D* **56** (1997) 7052;
M. Golterman and K. C. Leung, *Phys. Rev. D* **56** (1997) 2950;
Nucl. Phys. Proc. Suppl. **73** (1999) 246;
G. Colangelo and E. Pallante, *Nucl. Phys. B* **520** (1998) 433;
W. Bardeen, A. Duncan, E. Eichten and H. Thacker, hep-lat/0007010.
59. For an alternative approach, see
L. Lellouch and M. Lüscher, hep-lat/0003023, and the references therein.
60. T. Banks and A. Casher, *Nucl. Phys. B* **169** (1980) 103;
E. Marinari, G. Parisi and C. Rebbi, *Phys. Rev. Lett.* **47** (1981) 1795.
61. H. Leutwyler and A. Smilga, *Phys. Rev. D* **46** (1992) 5607;
A. Smilga and J. Stern, *Phys. Lett. B* **318** (1993) 531;
J. C. Osborn, D. Toublan and J. J. Verbaarschot,
Nucl. Phys. B **540** (1999) 317;
D. Toublan and J. J. Verbaarschot, *Nucl. Phys. B* **560** (1999) 259;
K. Zyablyuk, *JHEP* **0006** (2000) 025;
Taro Nagao and Shinsuke M. Nishigaki, hep-th/0001137;

- J. B. Kogut, M. A. Stephanov and D. Toublan,
Phys. Lett. B **464** (1999) 183;
 J. B. Kogut, M. A. Stephanov, D. Toublan, J. J. Verbaarschot and
 A. Zhitnitsky, *Nucl. Phys. B* **582** (2000) 477.
62. E. Witten, *Nucl. Phys. B* **156** (1979) 269;
 G. Veneziano, *Nucl. Phys. B* **159** (1979) 213.
63. R. Kaiser and H. Leutwyler, in *Nonperturbative Methods in Quantum
 Field Theory*, eds. A. W. Schreiber, A. G. Williams and A. W. Thomas
 (World Scientific, Singapore, 1998);
 E. P. Venugopal and B. R. Holstein, *Phys. Rev. D* **57** (1998) 4397;
 T. Feldmann, P. Kroll and B. Stech, *Phys. Rev. D* **58** (1998) 114006;
Phys. Lett. B **449** (1999) 339;
 B. Bagchi, P. Bhattacharyya, S. Sen and J. Chakrabarti,
Phys. Rev. D **60** (1999) 074002;
 F. Cao and A. I. Signal, *Phys. Rev. D* **60** (1999) 114012;
 L. S. Celenza, B. Huang and C. M. Shakin, *Phys. Rev. C* **59** (1999) 2814;
 R. Escribano and J. M. Frere, *Phys. Lett. B* **459** (1999) 288;
 T. Feldmann, *Int. J. Mod. Phys. A* **15** (2000) 159;
 B. Bagchi and P. Bhattacharyya, *Mod. Phys. Lett. A* **15** (2000) 167.
64. D. B. Kaplan and A. V. Manohar, *Phys. Rev. Lett.* **56** (1986) 2004.
65. B. Ananthanarayan, G. Colangelo, J. Gasser and H. Leutwyler,
 hep-ph/0005297. For recent work on the mathematical properties of the
 Roy equations, see:
 J. Gasser and G. Wanders, *Eur. Phys. J. C* **10** (1999) 159;
 G. Wanders, hep-ph/0005042.
66. I. Caprini, *Eur. Phys. J. C* **13** (2000) 471;
 M. Jamin, J. A. Oller and A. Pich, hep-ph/0006045.
67. G. J. Gounaris and J. J. Sakurai, *Phys. Rev. Lett.* **21** (1968) 244.
68. T. N. Truong, *Phys. Rev. D* **61** (1988) 2526; hep-ph/9809476;
 hep-ph/0001271; hep-ph/0006302;
 J. A. Oller, E. Oset and J. R. Pelaez, *Phys. Rev. Lett.* **80** (1998) 3452;
Phys. Rev. D **59** (1999) 074001;
 J. A. Oller, E. Oset and A. Ramos,
Prog. Part. Nucl. Phys. **45** (2000) 157.
69. The consistency of the chiral representation for the $\pi\pi$ scattering ampli-
 tude with the constraints imposed by analyticity, unitarity and crossing
 symmetry are discussed in:
 B. Ananthanarayan, D. Toublan and G. Wanders,
Phys. Rev. D **51** (1995) 1093; *ibid.* **53** (1996) 2362;

- G. Wanders, *Helv. Phys. Acta* **70** (1997) 287;
Phys. Rev. D **56** (1997) 4328.
70. G. Colangelo, J. Gasser and H. Leutwyler, hep-ph/0007112.
71. The numerical results for the scattering lengths quoted by Bijmens *et al.*³⁶ are based on a matching at threshold, where the chiral perturbation series converges less rapidly. The input used to determine the relevant effective coupling constants is discussed in:
 J. Nieves and E. Ruiz Arriola, hep-ph/9906437;
 G. Amoros, J. Bijmens and P. Talavera, *Phys. Lett. B* **480** (2000) 71;
 hep-ph/0003258.
72. S. Pislak *et al.*, “A new measurement of $K^+ \rightarrow \pi^+\pi^-e^+\nu$ ”, talk at Laboratori Nazionali di Frascati, June 22, 2000.
73. B. Adeva *et al.*, Proposal to the SPSLC, CERN/SPSLC 95-1 (1995);
 A. Lanaro *et al.* [DIRAC collaboration], *πN Newsletter* **15** (1999) 270.
74. J. Gasser, V. E. Lyubovitskij and A. Rusetsky,
Phys. Lett. B **471** (1999) 244.
75. N. H. Fuchs, H. Sazdjian and J. Stern, *Phys. Lett. B* **269** (1991) 183;
 J. Stern, H. Sazdjian and N. H. Fuchs, *Phys. Rev. D* **47** (1993) 3814;
 M. Knecht, B. Moussallam, J. Stern and N. H. Fuchs,
Nucl. Phys. B **457** (1995) 513; *ibid.* **471** (1996) 445.